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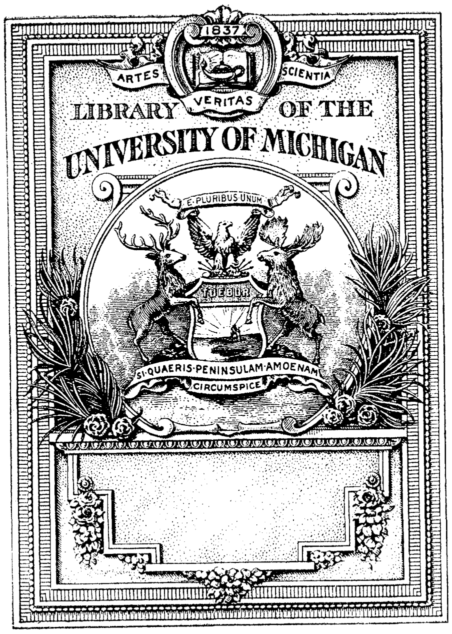
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A TREATISE ON TRIGONOMETRY.

A

TREATISE

ON

TRIGONOMETRY

BY

PROFS. OLIVER, WAIT AND JONES

OF

CORNELL UNIVERSITY.

NEW YORK:
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PREFACE.

THIS book is one of a series of text-books to be prepared by the department of mathematics of Cornell University, in accordance with the scheme of instruction now in force here. It was outlined and written mainly by Prof. Jones; but it has been carefully read by all of us, the general plan, and all difficulties, have been discussed together, the proofs have been submitted to all, and it goes out as our joint production. It is designed as a drill-book for class use; its leading features are:

The general definition of the trigonometric functions in terms applicable to all angles, without regard to sign or magnitude.

The expression of the functions of all angles in terms of the functions of positive angles less than a right angle, by direct reference to the definitions.

The graphical representation of functions.

The general proof of the formulae for the functions of the sum and difference of two angles, of double angles, half-angles, etc.

The differentiation of trigonometric functions, their development thereby into series, and the computation of the trigonometric canon by means of these series.

The solution of oblique triangles by means of right triangles, as well as by the general properties of triangles; and by the use of natural as well as logarithmic functions.

An exhaustive discussion of the ambiguous and impossible cases of right and oblique triangles.

A careful choice and arrangement of topics, according to their relations to practical work and to the higher mathematics.

The exact statement of principles in the form of theorems and corollaries, and their rigorous demonstration.

Frequent reference of collateral matter to the reader for demonstration.

Copious and varied exercises.

In the preparation of the book, free use has been made of the works of other authors, particularly those of Briot and Bouquet, De Morgan, Todhunter, Peirce, Wheeler, Greenleaf, Loomis, and Chauvenet; and of collections of exercises representing Senate-house examination papers.

The careful reader will doubtless find many typographical and other errors in this first edition; he will confer a great favor if he will kindly communicate them to either of the authors. Any suggestions from practical teachers, looking to the improvement of the book in either matter or form, will be welcomed and esteemed of great value.

Among other such improvements now in contemplation is the addition of a chapter on the applications of spherical trigonometry to astronomy, geodesy and navigation, and one on imaginaries, and an alphabetical index to the whole.

The starred portions of the book may be omitted without breaking the continuity of the subject; and to such teachers as do not desire to take up the whole treatise the following abridgment is recommended:

I. §§ 1–23, except the note to § 18, and Note 4 to § 19; selections from Ex. 1–7, 9, 21–23, and 25–28.
II. §§ 1–3.
III. § 2; § 3, one method, and § 1 if the *second method* is chosen; Ex. 1–19 and 26–45.
IV. §§ 1, 2, 4, 5 and 6, one method, and § 3 if the *second method* is chosen, except Thms. 8–10; Ex. 1–26.

O. W. J.

ITHACA, N.Y., *April* 8, 1881.

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TRIGONOMETRY.

TRIGONOMETRY is that branch of mathematics which treats of the numerical relations of angles and triangles. It is essentially algebraic in character, but is founded on Geometry. It has two parts: *Plane Trigonometry*, which treats of plane angles and triangles, and *Spherical Trigonometry*, which treats of spherical angles and triangles.

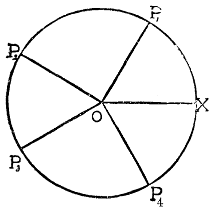
PLANE TRIGONOMETRY.

I. PRIMARY DEFINITIONS AND FORMULAE.

§ 1. PLANE ANGLES.

A *plane angle* is the opening between two straight lines which meet at a point, and is generated by revolving one of the lines about the common point as a hinge. The fixed line is the *initial line,* and the moving line is the *terminal line*. The revolving line has made one revolution when, by a continuous forward motion, it has come again to the position it first occupied. An angle is not limited in Trigonometry by the geometrical words "acute" and "obtuse"; it may exceed a half revolution, or a whole revolution, or it may be any number of revolutions, thus:

If the line OP first coincide with OX, and then, revolving about O, take successively the positions OP_1, OP_2, OP_3, OP_4, OP_1, ·····, the angles generated are XOP_1, XOP_2, XOP_3, XOP_4, ·····, and these angles, by general agreement, are called positive angles,



while XOP_4, XOP_3, ····· are negative angles. That is to say, revolution from right to left, and opposite to the hands of a clock, is *positive revolution;* and revolution from left to right is *negative revolution.* The order of the letters indicates whether the angle is positive or negative, thus:

XOP_1 is positive, and P_1OX is negative;

$XOP_1 + P_1OX = 0$,

$XOP_1 + P_1OP_2 + P_2OX = 0$,

$XOP_4 + P_4OP_3 + P_3OX = 0$; . . .

but $P_1OP_2 + P_2OP_3 + P_3OP_4 + P_4OP_1 =$ one entire positive revolution,

and $P_1OP_4 + P_4OP_3 + P_3OP_2 + P_2OP_1 =$ one entire negative revolution.

So, the arcs generated by a point P, moving along a circle, are positive or negative arcs, according as the movement of the point indicates positive or negative revolution of the radius OP, and therefore, according as the angles which they subtend are positive or negative angles, thus:

XP_1, XP_2, ····· are *positive arcs,*

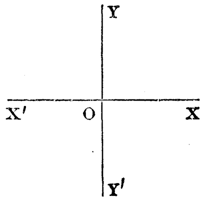
but XP_4, XP_3, ····· are *negative arcs;*

and $XP_1 + P_1P_2 + P_2P_3 + P_3P_2 + P_2X = 0$,

but $XP_1 + P_1P_2 + P_2P_3 + P_3P_4 + P_4X =$ one circumference.

If $Y'Y \perp X'X$ at O; then:

XOY is the *first quadrant,*
YOX' is the *second quadrant,*
$X'OY'$ is the *third quadrant,*
$Y'OX$ is the *fourth quadrant.*



An angle is said to be in the first, second, third, or fourth quadrant, according as its terminal line lies in the first, second, third, or fourth quadrant, counting from the initial line.

If the revolution of the terminal line is continuous, it comes again and again to the same positions which it had during the first revolution; such returns are *periodic*.

§ 2. ANGULAR NOTATION.

The right angle is divided into 90 equal parts called *degrees*, the degree, into 60 equal parts called *minutes;* the minute, into 60 equal parts called *seconds*. Degrees, minutes and seconds are marked $^\circ$, $'$, $''$, respectively, thus:

$6^\circ\,29'\,33''.3$ is read 6 degrees, 29 minutes, 33 and 3 tenths seconds.

When angles are expressed in degrees, minutes and seconds, they are said to be expressed in *degree-measure*.

Another measure called *circular measure*, *arcual measure*, or, briefly, *π-measure*, comes from the use of the ratio of an arc which subtends an angle, to the radius of the arc;
for, since, in the same or equal circles, angles at the centre are proportional to the arcs which subtend them, and, in different circles, like arcs are proportional to their radii, [*geom.*
therefore the ratio, *arc* : *radius*, is proportional to the angle subtended, and may be used as a representative or measure of it.
But the ratio, *half-circle* : *radius*, equals 3.14159 ....., called π;
therefore a half-revolution, or an angle of 180°, is represented by π; a right angle, or 90°, by $\frac{\pi}{2}$; a whole revolution, or 360°, by 2π; 30°, by $\frac{\pi}{6}$; and so on.

That angle whose arc is as long as the radius, and which may be called the unit angle of the π-measure, is $\frac{180^\circ}{\pi}$, $= 57^\circ\,17'\,44''.8$.

The π-measure of an angle of 1° is $\frac{\pi}{180}$, $= .017\,4533$;
that of $1'$ is $.017\,4533 \div 60$, $= .000\,2909$;
that of $1''$ is $.000\,2909 \div 60$, $= .000\,0048$.

An angle given in π-measure may be expressed in degree-measure by multiplying it into the unit angle of π-measure;

and an angle given in degree-measure may be expressed in π-measure by multiplying the number of degrees, minutes or seconds by the π-measure of 1°, $1'$ or $1''$.

NOTE. The reader must carefully distinguish between the two notations, for though 90° and $\frac{\pi}{2}$ both represent a right angle, it is hardly right to say $90^\circ = \frac{\pi}{2}$, for 90° is a right angle, an actual geometrical magnitude, while $\frac{\pi}{2}$ is merely a ratio, *i.e.*, a number. With this caution, however, the angle, the arc, and the ratio, *arc* : *radius*, may be used almost at pleasure, and the one notation or the other may be employed at convenience. The π-measure is generally preferable for theoretical work, but the degree-measure for computation of triangles.

§ 3. COMPLEMENT AND SUPPLEMENT OF AN ANGLE OR ARC.

The *complement* of an angle is its defect from a right angle; of an arc, its defect from a quadrant; *i.e.*, it is the remainder when, from a right angle or quadrant, the given angle or arc is subtracted.

The *supplement* of an angle is its defect from two right angles; of an arc, its defect from a half-circle; *i.e.*, it is the remainder when, from two right angles or a half-circle, the given angle or arc is subtracted.

Manifestly the complement of a positive angle or arc less than 90° is a positive angle or arc less than 90°; of a positive angle or arc greater than 90°, is a negative angle or arc; of a negative angle or arc, is a positive angle or arc greater than 90°.

So, the supplement of a positive angle or arc less than 180° is a positive angle or arc less than 180°; of a positive angle or arc greater than 180°, is a negative angle or arc; of a negative angle or arc, is a positive angle or arc greater than 180°. Thus:

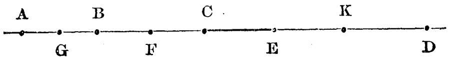
the complement of 75° is 15°; of 100°, is -10°; of -10°, is 100°; of $\frac{\pi}{6}$, is $\frac{\pi}{3}$; of 2π, is $-\frac{3\pi}{2}$;

the supplement of 75° is 105°; of 200°, is -20°; of -20°, is 200°; of $\frac{\pi}{6}$, is $\frac{5\pi}{6}$; of 2π, is $-\pi$.

§ 4. POSITIVE AND NEGATIVE LINES.

If two or more points lie on a straight line, and the position of any one of them be known, then the positions of the other points are determined by their distances and directions from the point first named. The first point is the *origin;* and, of the two directions from this point, along the line, if either be called positive, the other is negative, and the segments of the line measured in one direction are positive, in the other negative. The order of the letters at the extremities of a segment indicates the direction of the segment, thus:
if A and B be two points on a line, and AB be positive,
then BA is negative, and *vice versa;* and, in either case,

$$\mathrm{AB} + \mathrm{BA} = 0.$$



So, if A, B, C be any three points on a line, then, whatever their order upon the line,

$$\mathrm{AB} + \mathrm{BC} = \mathrm{AC},$$

and $$\mathrm{AB} + \mathrm{BC} + \mathrm{CA} = 0.$$

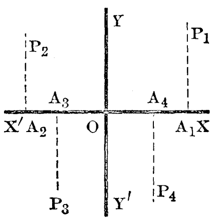
So, if A, B, C, D, ····· K be n points on a line, then, whatever their order upon the line,

$$\mathrm{AB} + \mathrm{BC} + \mathrm{CD} + \cdots\cdots = \mathrm{AK},$$

and $$\mathrm{AB} + \mathrm{BC} + \mathrm{CD} + \cdots\cdots + \mathrm{KA} = 0.$$

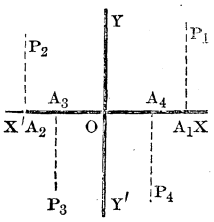
§ 5. POSITION OF A POINT IN A PLANE BY ABSCISSA AND ORDINATE.

Let P be any point in a plane, O a fixed point, and OX a fixed line in the plane; draw $\mathrm{AP} \perp \mathrm{OX}$; then the position of P with reference to O and OX is determined when the length and directions of OA and AP are known. In this figure:
the reference-point O is the *origin;*
the horizontal distance OA is the *abscissa*,
and the perpendicular AP is the *ordinate,* of P.



The line X′OX is the *axis of abscissas*,
and Y′OY is the *axis of ordinates*.

The abscissa and ordinate of a point, when spoken of together, are its *coördinates*.



By general agreement, when the line OX is so placed before the reader that X is its right extremity, then OX is assumed as the positive direction of the axis, and OX′ as its negative direction;
therefore abscissas of points in the first and fourth quadrants are positive, and abscissas of points in the second and third quadrants are negative.

By general agreement, also, that side of the axis from which positive revolution is had is assumed to be positive, and that side from which negative revolution is had is negative;
therefore ordinates of points in the first and second quadrants are positive, and ordinates of points in the third and fourth quadrants are negative.

The signs of abscissa and ordinate when written together are:

$$++, \quad -+, \quad --, \quad +-,$$

respectively, for points in the four quadrants taken in their order.

For brevity, the abscissa of a point may be denoted by x and the ordinate by y, and the position of a point is then determined from the equations

$$x = a \quad \text{and} \quad y = b,$$

wherein a, with its sign, stands for the length and direction of the abscissa of the point,
and b, with its sign, for the length and direction of its ordinate.

If $x = 0$, the point is on the axis of ordinates;
if $y = 0$, it is on the axis of abscissas;
if both x and $y = 0$, it is at the origin.

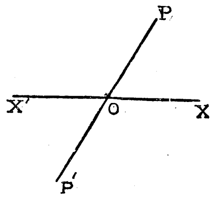
If r represents the distance between two points, P_1 and P_2;
then, the difference of their abscissas, $x_1 - x_2$,
and the difference of their ordinates, $y_1 - y_2$,
are the projections of r upon the two axes, respectively;
and $r^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$.

§ 6. POSITION OF A POINT BY BEARING AND DISTANCE.

When two straight lines meet, the point of intersection is the origin; one of the lines is the initial line, the other is the terminal line, and the angle between them is the *bearing*, from the initial line, of any point on the terminal line.

If, then, the distance of any point from the origin, and its bearing from the initial line, be known, the position of the point is determined.

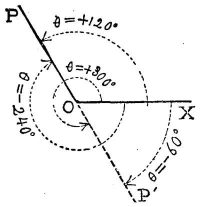
The positive directions of the lines which enclose an angle, taken with reference to that angle, are from the origin, *i.e.*, from the vertex of the angle, measured along the sides of the angle; thus, with reference to XOP,
the sides OX and OP are positive,
and OX′ and OP′ are negative.
But, with reference to X′OP′,
the sides OX′ and OP′ are positive,
and OX and OP are negative.
So, with reference to POX′ or to X′OP,
the sides OP and OX′ are positive,
and OP′ and OX are negative.



Since the terminal line may revolve to the right or to the left, and since the distance may be measured in either the positive or the negative direction, it is manifest that a point may be determined by four different combinations of bearing and distance.

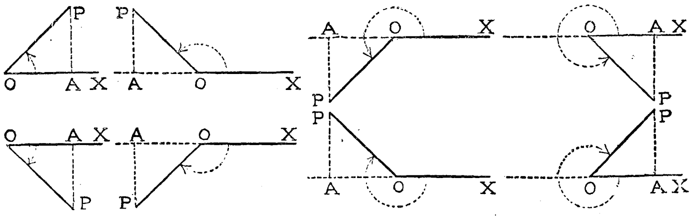
Let r stand for the distance and θ for the bearing of the point P; then either:

$r = {}^{+}\text{OP}$, and $\theta = {}^{+}\text{XOP}$;
or $r = {}^{+}\text{OP}$, and $\theta = {}^{-}\text{XOP}$;
or $r = {}^{-}\text{OP}$, and $\theta = {}^{+}\text{XOP}'$;
or $r = {}^{-}\text{OP}$, and $\theta = {}^{-}\text{XOP}'$;



wherein the signs ⁺ and ⁻ are signs of quality, and not of operation; they show whether the line OP and the angle XOP are taken in the positive or the negative direction. For the purposes of this treatise the first two combinations are sufficient.

§ 7. TRIGONOMETRIC FUNCTIONS.



Let XOP be any angle, positive or negative, OX the initial line, OP the terminal line, and P any point upon it. Draw AP $\perp$ OX; then OP is the *distance* of the point P, OA is its abscissa, AP is its ordinate, and the six ratios between the three lines r, x, and y are the *trigonometric functions* of the angle XOP, viz.:

The ratio	*i.e.*,	is the	written
ordinate : distance	$y : r$	*sine* of XOP	sin XOP
abscissa : distance	$x : r$	*cosine* of XOP	cos XOP
ordinate : abscissa	$y : x$	*tangent* of XOP	tan XOP
abscissa : ordinate	$x : y$	*cotangent* of XOP	cot XOP
distance : abscissa	$r : x$	*secant* of XOP	sec XOP
distance : ordinate	$r : y$	*cosecant* of XOP	csc XOP

Two subsidiary functions, sometimes used, are the *versed sine* and *coversed sine;* their definitions are:

$$\text{vers XOP} = 1 - \cos \text{XOP}, \quad \text{covers XOP} = 1 - \sin \text{XOP}.$$

NOTE. From these definitions result directly the six equations:

ordinate = distance . sin XOP; distance = ordinate . csc XOP;
abscissa = distance . cos XOP; distance = abscissa . sec XOP;
ordinate = abscissa . tan XOP; abscissa = ordinate . cot XOP.

The sine and cosine may thus be called *projecting factors*, since their effect as multipliers is to project the distance upon the axes of ordinates and of abscissas respectively. So, the tangent and cotangent may be called *interchanging factors*, since their effect is to convert abscissas into ordinates, and *vice versa*.

§ 8. ANTI-FUNCTIONS.

The expressions $\sin^{-1}a$, $\cos^{-1}a$, $\tan^{-1}a$, ….. are called *anti-functions*, and are read the *anti-sine* of a, the *anti-cosine* of a, ….. They mean "the angle whose sine is a," "the angle whose cosine is a," and so on; thus:

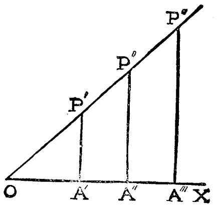
if $\quad a = \sin\theta$, $\quad$ then $\quad \theta = \sin^{-1}a$;

if $\quad b = \cos\theta$, $\quad$ then $\quad \theta = \cos^{-1}b$; …..

§ 9. CONSTANCY OF FUNCTIONS.

THEOREM 1. *The functions of a given angle are constant, whatever points* P′, P″, ….. *are taken on the terminal line.*

For, from P′, P″, ….. draw A′P′, A″P″, ….. ⊥ OX; then the triangles OA′P′, OA″P″, ….. are similar, and the ratios of their homologous sides are equal, viz.:



$$y' : r' = y'' : r'', \ = \cdots$$
$$x' : r' = x'' : r'', \ = \cdots$$
$$y' : x' = y'' : x'', \ = \cdots$$

Q. E. D. [*geom.*

§ 10. PERIODICITY OF FUNCTIONS.

THM. 2. *The functions of any angle* θ *and of* $2n\pi + \theta$ (n being any integer, positive or negative) *are identical; sine with sine, cosine with cosine, and so on.*

For, ∵ $\pm 2\pi$, $\pm 4\pi$, ….. $\pm 2n\pi$ stand for one, two, ….. n entire revolutions, forward or backward,

∴ OP has the same position for the angles θ, $\pm 2\pi + \theta$, $\pm 4\pi + \theta$, ….. $\pm 2n\pi + \theta$; and r, x and y are identical, each with each, for the angles thus formed.

∴ the ratios are identical; sine with sine, cosine with cosine, and so on. Q. E. D.

COROLLARY 1. The functions of any negative angle, $-\theta$, and of the positive angle, $2\pi - \theta$, which has the same bounding lines, are identical.

NOTE. The trigonometric functions are therefore *periodic functions* of angles, and Trigonometry is sometimes defined as "that branch of Algebra which treats of periodic functions." If $\theta = \sin^{-1}a$, then θ is any one of an infinite number of angles all of which have the same sine, a.

§ 11. SIGNS OF FUNCTIONS.

THM. 3. *The signs of functions of angles in the four quadrants are:*

Angle in	sin and csc	cos and sec	tan and cot	vers and covers
First quadrant	+	+	+	+
Second quadrant	+	−	−	+
Third quadrant	−	−	+	+
Fourth quadrant	−	+	−	+

For, ∵ r is measured along the side of the angle, and is therefore always positive; [§ 6
and ∵ x is positive in the first and fourth quadrants, and negative in the second and third; [§ 5
and ∵ y is positive in the first and second quadrants, and negative in the third and fourth; [§ 5
∴ the signs of the several ratios are as given above.

§ 12. FUNCTIONS OF NEGATIVE ANGLES.

THM. 4. *For any negative angle, $-\theta$, the values of the several functions, in terms of the functions of the opposite positive angle, $+\theta$, are:*

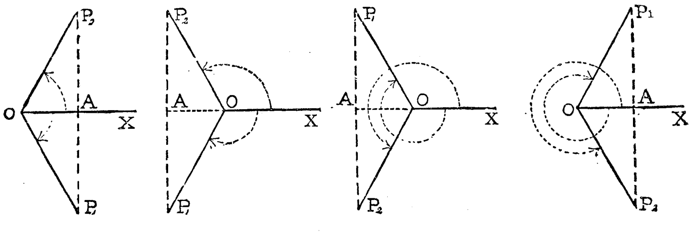
1] $\sin(-\theta) = -\sin\theta$; $\csc(-\theta) = -\csc\theta$;

2] $\cos(-\theta) = +\cos\theta$; $\sec(-\theta) = +\sec\theta$;

3] $\tan(-\theta) = -\tan\theta$; $\cot(-\theta) = -\cot\theta$;

For, let XOP_1 be any negative angle, $-\theta$, and about the initial line OX, as an axis of symmetry, draw OP_2, making XOP_2 *opposite* to XOP_1; *i.e.*, of equal magnitude, but of opposite sign.

Take OP_1 and OP_2 equal distances on the terminal lines, and join P_1P_2; then, since OX bisects P_1P_2 at right angles, [*geom.*



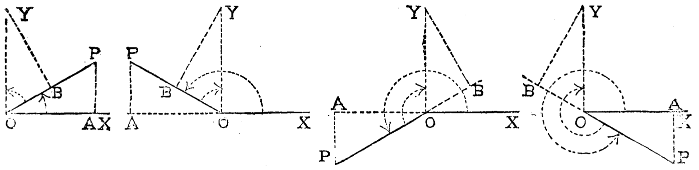
therefore the abscissas, x_1 and x_2, of P_1 and P_2 are identical, and their ordinates y_1 and y_2 are opposites;

$$\therefore \quad \frac{y_1}{r_1} = -\frac{y_2}{r_2}, \quad \frac{x_1}{r_1} = \frac{x_2}{r_2}, \quad \frac{y_1}{x_1} = -\frac{y_2}{x_2}, \quad \ldots\ldots \qquad \text{Q. E. D.}$$

§ 13. FUNCTIONS OF $\frac{1}{2}\pi - \theta$, THE COMPLEMENT OF AN ANGLE θ.

THM. 5. *Any function of the complement of an angle, co-θ or $\frac{1}{2}\pi - \theta$, is the co-function of the angle*, i.e.:

4] $\sin \text{co-}\theta = \cos\theta$; $\csc \text{co-}\theta = \sec\theta$;
5] $\cos \text{co-}\theta = \sin\theta$; $\sec \text{co-}\theta = \csc\theta$;
6] $\tan \text{co-}\theta = \cot\theta$; $\cot \text{co-}\theta = \tan\theta$.



For, let XOP be any angle θ, and POY its complement, $\frac{1}{2}\pi - \theta$. Draw AP, ordinate of P with reference to OX; take OY = OP, and draw BY, ordinate of Y with reference to OP; then,

$\because \quad \triangle \text{YBO} = \triangle \text{OAP}$, [*geom.*
$\therefore \quad \text{OB} = \text{AP}, \quad \text{and BY} = \text{OA}; \quad \text{and OY} = \text{OP};$ [*constr.*
i.e., $x' = y, \quad y' = x, \quad \text{and} \quad r' = r.$
$\therefore \quad \sin \text{POY} = y' : r' = x : r = \cos \text{XOP};$
$\cos \text{POY} = x' : r' = y : r = \sin \text{XOP};$

and so for the other functions, as the reader may prove.

NOTE. The words cosine, cotangent, and cosecant are abbreviated forms for complement-sine, complement-tangent, and complement-secant; *i.e.*, for sine of complement, tangent of complement, and secant of complement.

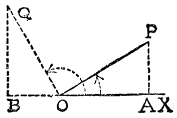
§ 14. FUNCTIONS OF $\frac{1}{2}\pi+\theta$.

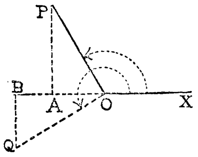
THM. 6. *For any angle, $\frac{1}{2}\pi+\theta$, the values of the several functions, in terms of the functions of θ, are:*

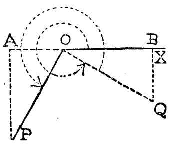
7] $\sin(\frac{1}{2}\pi+\theta)=+\cos\theta$; $\csc(\frac{1}{2}\pi+\theta)=+\sec\theta$;

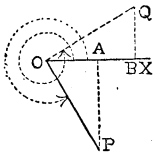
8] $\cos(\frac{1}{2}\pi+\theta)=-\sin\theta$; $\sec(\frac{1}{2}\pi+\theta)=-\csc\theta$;

9] $\tan(\frac{1}{2}\pi+\theta)=-\cot\theta$; $\cot(\frac{1}{2}\pi+\theta)=-\tan\theta$.









For, let XOP be any angle θ, and $\text{POQ}=\frac{1}{2}\pi$; then $\text{XOQ}=\frac{1}{2}\pi+\theta$. Take OQ = OP, and draw AP and BQ ordinates of P and Q with reference to OX; then,

$\because$ $\triangle\,\text{QBO}=\triangle\,\text{OAP}$, [*geom.*

$\therefore$ $\text{OB}=-\text{AP}$, and $\text{BQ}=\text{OA}$; and $\text{OQ}=\text{OP}$; [*constr.*

i.e., $x'=-y$, $y'=x$, and $r'=r$.

$\therefore$ $\sin\text{XOQ}=y':r'=x:r=\cos\text{XOP}$;

$\cos\text{XOQ}=x':r'=-y:r=-\sin\text{XOP}$;

and so on, as the reader may prove.

§ 15. FUNCTIONS OF $\pi-\theta$, THE SUPPLEMENT OF AN ANGLE θ.

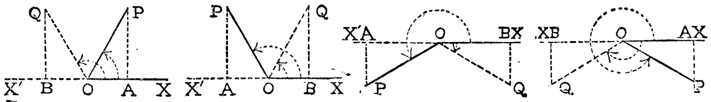
THM. 7. *For the supplement of any angle, $\pi-\theta$, the values of the several functions, in terms of the functions of θ, are:*

10] $\sin(\pi-\theta)=+\sin\theta$; $\csc(\pi-\theta)=+\csc\theta$;

11] $\cos(\pi-\theta)=-\cos\theta$; $\sec(\pi-\theta)=-\sec\theta$;

12] $\tan(\pi-\theta)=-\tan\theta$; $\cot(\pi-\theta)=-\cot\theta$.

For, let XOP be any angle θ, and draw OQ so that $\angle$ QOX$'$ = XOP; then $\angle$ XOQ $= \pi - \theta$.



Take OQ = OP, and draw AP and BQ, ordinates of P and Q with reference to OX; then,

$\because$ $\triangle$ OBQ = $\triangle$ OAP, [*geom.*

$\therefore$ OB = $-$ OA, and BQ = AP; and OQ = OP; [*constr.*

i.e., $x' = -x$, $y' = y$, and $r' = r$.

$\therefore$ $\sin$ XOQ $= y' : r' = y : r = \sin$ XOP;

$\cos$ XOQ $= x' : r' = -x : r = -\cos$ XOP;

and so on, as the reader may prove.

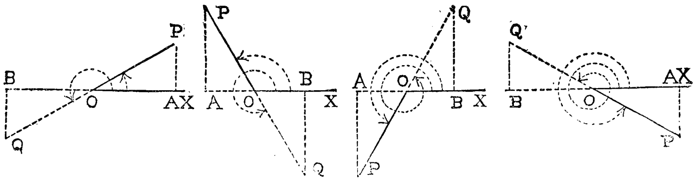
§ 16. FUNCTIONS OF $\pi + \theta$.

THM. 8. *For any angle, $\pi + \theta$, the values of the several functions, in terms of the functions of θ, are:*

13] $\sin(\pi + \theta) = -\sin\theta$; $\csc(\pi + \theta) = -\csc\theta$;

14] $\cos(\pi + \theta) = -\cos\theta$; $\sec(\pi + \theta) = -\sec\theta$;

15] $\tan(\pi + \theta) = +\tan\theta$; $\cot(\pi + \theta) = +\cot\theta$.



For, let XOP be any angle θ, and produce PO; then XOQ $= \pi + \theta$. Take OQ = OP, and draw AP and BQ ordinates of P and Q with reference to OX; then,

$\because$ $\triangle$ OBQ = $\triangle$ OAP, [*geom.*

$\therefore$ OB = $-$OA, and BQ = $-$ AP; and OQ = OP; [*constr.*

i.e., $x' = -x$, $y' = -y$, and $r' = r$.

$\therefore$ etc., as the reader may prove.

§ 17. FUNCTIONS OF $\frac{3\pi}{2}-\theta$.

THM. 9. *For any angle,* $\frac{3\pi}{2}-\theta$, *the values of the several functions, in terms of the functions of* θ, *are:*

16] $\sin\left(\frac{3\pi}{2}-\theta\right)=-\cos\theta$; $\csc\left(\frac{3\pi}{2}-\theta\right)=-\sec\theta$;

17] $\cos\left(\frac{3\pi}{2}-\theta\right)=-\sin\theta$; $\sec\left(\frac{3\pi}{2}-\theta\right)=-\csc\theta$;

18] $\tan\left(\frac{3\pi}{2}-\theta\right)=+\cot\theta$; $\cot\left(\frac{3\pi}{2}-\theta\right)=+\tan\theta$.

The reader may prove.

§ 18. FUNCTIONS OF $\frac{3\pi}{2}+\theta$.

THM. 10. *For any angle,* $\frac{3\pi}{2}+\theta$, *the values of the several functions, in terms of the functions of* θ, *are:*

19] $\sin\left(\frac{3\pi}{2}+\theta\right)=-\cos\theta$; $\csc\left(\frac{3\pi}{2}+\theta\right)=-\sec\theta$;

20] $\cos\left(\frac{3\pi}{2}+\theta\right)=+\sin\theta$; $\sec\left(\frac{3\pi}{2}+\theta\right)=+\csc\theta$;

21] $\tan\left(\frac{3\pi}{2}+\theta\right)=-\cot\theta$; $\cot\left(\frac{3\pi}{2}+\theta\right)=-\tan\theta$.

The reader may prove.

*NOTE. The ten theorems just proved may be summarized and generalized as follows:

22] $$\left.\begin{matrix}\sin\\ \csc\end{matrix}\right\}\theta=\left.\begin{matrix}\sin\\ \csc\end{matrix}\right\}[n\pi+(-1)^n\cdot\theta]=\left.\begin{matrix}\cos\\ \sec\end{matrix}\right\}[(n+\tfrac{1}{2})\pi-(-1)^n\cdot\theta]$$
$$=\left.\begin{matrix}-\sin\\ -\csc\end{matrix}\right\}[n\pi-(-1)^n\cdot\theta]=\left.\begin{matrix}-\cos\\ -\sec\end{matrix}\right\}[(n+\tfrac{1}{2})\pi+(-1)^n\cdot\theta];$$

23] $$\left.\begin{matrix}\cos\\ \sec\end{matrix}\right\}\theta=\left.\begin{matrix}\cos\\ \sec\end{matrix}\right\}[2n\pi\pm\theta]=\left.\begin{matrix}\sin\\ \csc\end{matrix}\right\}[(2n+\tfrac{1}{2})\pi\pm\theta]$$
$$=\left.\begin{matrix}-\cos\\ -\sec\end{matrix}\right\}[(2n+1)\pi\pm\theta]=\left.\begin{matrix}-\sin\\ -\csc\end{matrix}\right\}[(2n-\tfrac{1}{2})\pi\pm\theta];$$

24] $$\left.\begin{matrix}\tan\\ \cot\end{matrix}\right\}\theta=\left.\begin{matrix}\tan\\ \cot\end{matrix}\right\}[n\pi+\theta]=\left.\begin{matrix}\cot\\ \tan\end{matrix}\right\}[(n+\tfrac{1}{2})\pi-\theta]$$
$$=\left.\begin{matrix}-\tan\\ -\cot\end{matrix}\right\}[n\pi-\theta]=\left.\begin{matrix}-\cot\\ -\tan\end{matrix}\right\}[(n-\tfrac{1}{2})\pi+\theta].$$

They also give the following:

25] $\sin^{-1}(\sin\theta)=\csc^{-1}(\csc\theta)=2n\pi+\theta$ *or* $(2n+1)\pi-\theta$;
26] $\cos^{-1}(\cos\theta)=\sec^{-1}(\sec\theta)=2n\pi\pm\theta$;
27] $\tan^{-1}(\tan\theta)=\cot^{-1}(\cot\theta)=n\pi+\theta$.

§ 19. VALUES OF FUNCTIONS IN TERMS OF EACH OTHER.

THM. 11. *For any angle* θ, *the values of the several functions, in terms of each other, are:*

[28] $\sin\theta=$	[29] $\cos\theta=$	[30] $\tan\theta=$	[31] $\cot\theta=$	[32) $\sec\theta=$	[33] $\csc\theta=$
$\sin\theta$	$\sqrt{1-\sin^2\theta}$	$\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$	$\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}$	$\frac{1}{\sqrt{1-\sin^2\theta}}$	$\frac{1}{\sin\theta}$
$\sqrt{1-\cos^2\theta}$	$\cos\theta$	$\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}$	$\frac{\cos\theta}{\sqrt{1-\cos^2\theta}}$	$\frac{1}{\cos\theta}$	$\frac{1}{\sqrt{1-\cos^2\theta}}$
$\frac{\tan\theta}{\sqrt{\tan^2\theta+1}}$	$\frac{1}{\sqrt{\tan^2\theta+1}}$	$\tan\theta$	$\frac{1}{\tan\theta}$	$\sqrt{\tan^2\theta+1}$	$\frac{\sqrt{\tan^2\theta+1}}{\tan\theta}$
$\frac{1}{\sqrt{\cot^2\theta+1}}$	$\frac{\cot\theta}{\sqrt{\cot^2\theta+1}}$	$\frac{1}{\cot\theta}$	$\cot\theta$	$\frac{\sqrt{\cot^2\theta+1}}{\cot\theta}$	$\sqrt{\cot^2\theta+1}$
$\frac{\sqrt{\sec^2\theta-1}}{\sec\theta}$	$\frac{1}{\sec\theta}$	$\sqrt{\sec^2\theta-1}$	$\frac{1}{\sqrt{\sec^2\theta-1}}$	$\sec\theta$	$\frac{\sec\theta}{\sqrt{\sec^2\theta-1}}$
$\frac{1}{\csc\theta}$	$\frac{\sqrt{\csc^2\theta-1}}{\csc\theta}$	$\frac{1}{\sqrt{\csc^2\theta-1}}$	$\sqrt{\csc^2\theta-1}$	$\frac{\csc\theta}{\sqrt{\csc^2\theta-1}}$	$\csc\theta$

For $\because$ $\sin\theta=y:r$, $\cos\theta=x:r$, $\tan\theta=y:x$,
and $\because$ $\csc\theta=r:y$, $\sec\theta=r:x$, $\cot\theta=x:y$, [§ 7
34] $\therefore$ $\sin\theta\cdot\csc\theta=1$, $\cos\theta\cdot\sec\theta=1$, $\tan\theta\cdot\cot\theta=1$;
35] and $\tan\theta=\frac{\sin\theta}{\cos\theta}$, and $\cot\theta=\frac{\cos\theta}{\sin\theta}$;
i.e., sine and cosecant are reciprocals,
cosine and secant are reciprocals,
and tangent and cotangent are reciprocals;

whatever value is found for $\sin\theta$, its reciprocal is a value of $\csc\theta$, whatever value is found for $\cos\theta$, its reciprocal is a value of $\sec\theta$, whatever value is found for $\tan\theta$, its reciprocal is a value of $\cot\theta$, and whatever values are found for $\sin\theta$ and $\cos\theta$, their ratios are values of $\tan\theta$ and $\cot\theta$.

So, $\because\quad x^2+y^2=r^2,$ [*geom.*

$\therefore$ (1) $\dfrac{x^2}{r^2}+\dfrac{y^2}{r^2}=1,$

36] *i.e.*, $\sin^2\theta+\cos^2\theta=1;$

$\therefore\quad \sin\theta=\sqrt{1-\cos^2\theta},\quad$ and $\cos\theta=\sqrt{1-\sin^2\theta};$

and (2) $1+\dfrac{y^2}{x^2}=\dfrac{r^2}{x^2},$

37] *i.e.*, $1+\tan^2\theta=\sec^2\theta;$

$\therefore\quad \tan\theta=\sqrt{\sec^2\theta-1},\quad$ and $\sec\theta=\sqrt{1+\tan^2\theta};$

and (3) $\dfrac{x^2}{y^2}+1=\dfrac{r^2}{y^2};$

38] *i.e.*, $\cot^2\theta+1=\csc^2\theta;$

$\therefore\quad \cot\theta=\sqrt{\csc^2\theta-1},\quad$ and $\csc\theta=\sqrt{1+\cot^2\theta}.$

So, by combination of the formulae which have been proved, others are established; for example:

$$\cos\theta=\frac{1}{\sec\theta}=\frac{1}{\sqrt{1+\tan^2\theta}};$$

and $\because\quad \dfrac{\sin\theta}{\cos\theta}=\tan\theta,$

$\therefore\quad \sin\theta=\tan\theta\,.\cos\theta=\dfrac{\tan\theta}{\sqrt{1+\tan^2\theta}}.$

NOTE 1. These formulae, when taken two and two, are symmetrical, for example:

(1) Those for sine, in terms of cosine, tangent, secant, ... with those for cosine, in terms of sine, cotangent, cosecant, ...

(2) Those for tangent, in terms of sine, cosine, secant, ... with those for cotangent, in terms of cosine, sine, cosecant, ...

(3) Those for secant, in terms of sine, cosine, tangent, ... with those for cosecant, in terms of cosine, sine, cotangent, ...

Note 2. For a given $\left\{\begin{matrix}\text{sine}\\ \text{cosecant,}\end{matrix}\right.$ the $\left\{\begin{matrix}\text{cosecant}\\ \text{sine}\end{matrix}\right.$ has but one value;
for a given $\left\{\begin{matrix}\text{cosine}\\ \text{secant,}\end{matrix}\right.$ the $\left\{\begin{matrix}\text{secant}\\ \text{cosine}\end{matrix}\right.$ has but one value;
for a given $\left\{\begin{matrix}\text{tangent}\\ \text{cotangent,}\end{matrix}\right.$ the $\left\{\begin{matrix}\text{cotangent}\\ \text{tangent}\end{matrix}\right.$ has but one value;
but in every other case there are two corresponding values, the one positive, and the other negative, for every function.
This appears alike from the double signs of the radicals involved, and from the relations of the abscissas and ordinates of points in the several quadrants; thus:
with a given distance, to every abscissa correspond two ordinates;
therefore to every cosine correspond two sines.
So, to every ordinate correspond two abscissas;
therefore to every sine correspond two cosines; and so on.

Note 3. Of the relations given above, the most important are those between sine and cosecant, cosine and secant, tangent and cotangent;
those between sine and cosine, tangent and secant, cotangent and cosecant;
and the expressions for tangent and cotangent as ratios of sine and cosine, and for sine and cosine in terms of tangent.
These relations should be committed to memory; the rest may be proved as exercises.

* Note 4. The above relations may also be expressed thus:

$$\sin^{-1} a = \csc^{-1}\frac{1}{a},$$

$$\sin^{-1}(\pm a) = \cos^{-1}\sqrt{1-a^2} = \sec^{-1}\frac{1}{\sqrt{1-a^2}}$$

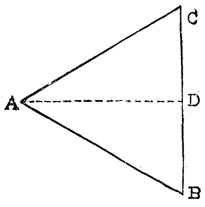
$$= \tan^{-1}\frac{\pm a}{\sqrt{1-a^2}} = \cot^{-1}\frac{\sqrt{1-a^2}}{\pm a},$$

wherein $\sqrt{1-a^2}$ is either positive or negative throughout, independently of $\pm a$. The reader may express in like manner the values of the five other principal anti-functions of a.

§ 20. FUNCTIONS OF 30°, 60°, 120°, 150°, ..., *i.e.*, OF $\frac{n\pi}{2} \pm \frac{\pi}{6}$.

THM. 12. *The functions of* $\frac{n\pi}{2} \pm \frac{\pi}{6}$ *are:*

Angle.	Sine.	Cos.	Tan.	Cot.	Sec.	Csc.
30°, *i.e.*, $\frac{\pi}{6}$; or $2n\pi + \frac{\pi}{6}$	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	$\sqrt{3}$	$\frac{2}{3}\sqrt{3}$	2
60°, *i.e.*, $\frac{2\pi}{6}$; or $(2n+\frac{1}{2})\pi - \frac{\pi}{6}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	2	$\frac{2}{3}\sqrt{3}$
120°, *i.e.*, $\frac{4\pi}{6}$; or $(2n+\frac{1}{2})\pi + \frac{\pi}{6}$	$\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	-2	$\frac{2}{3}\sqrt{3}$
150°, *i.e.*, $\frac{5\pi}{6}$; or $(2n+1)\pi - \frac{\pi}{6}$	$\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	$-\sqrt{3}$	$-\frac{2}{3}\sqrt{3}$	2
210°, *i.e.*, $\frac{7\pi}{6}$; or $(2n+1)\pi + \frac{\pi}{6}$	$-\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	$\sqrt{3}$	$-\frac{2}{3}\sqrt{3}$	-2
240°, *i.e.*, $\frac{8\pi}{6}$; or $(2n-\frac{1}{2})\pi - \frac{\pi}{6}$	$-\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	-2	$-\frac{2}{3}\sqrt{3}$
300°, *i.e.*, $\frac{10\pi}{6}$; or $(2n-\frac{1}{2})\pi + \frac{\pi}{6}$	$-\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	2	$-\frac{2}{3}\sqrt{3}$
330°, *i.e.*, $\frac{11\pi}{6}$; or $2n\pi - \frac{\pi}{6}$	$-\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	$-\sqrt{3}$	$\frac{2}{3}\sqrt{3}$	-2



For, let ABC be an equilateral triangle, and from A let fall AD $\perp$ BC; then AD bisects $\angle$ A, $= 60°$, and side BC; [*geom.*

$\therefore$ DAC $= 30°$, and DC $= \frac{1}{2}$AC.

But sin DAC $=$ DC : AC $= \frac{1}{2}$;

$\therefore \sin 30° = \frac{1}{2}$; $\qquad \therefore \cos 60° = \frac{1}{2}$;

$\therefore \cos 30° = \sqrt{1-\frac{1}{4}} = \frac{1}{2}\sqrt{3}$; $\qquad \therefore \sin 60° = \frac{1}{2}\sqrt{3}$.

The reader may prove the remaining values as exercises upon the preceding theorems; for example:

$$\tan\ 30° = \frac{\sin 30°}{\cos 30°} = \frac{\frac{1}{2}}{\frac{1}{2}\sqrt{3}} = \frac{1}{\sqrt{3}} = \tfrac{1}{3}\sqrt{3};$$

$$\sec\ 30° = \frac{1}{\cos 30°} = \frac{1}{\frac{1}{2}\sqrt{3}} = \frac{2}{\sqrt{3}} = \tfrac{2}{3}\sqrt{3};$$

$$\sin 150° = \sin(180° - 30°) = \sin 30° = \tfrac{1}{2};$$

$$\sec 330° = \sec(-30°) \qquad = \sec 30° = \tfrac{2}{3}\sqrt{3}.$$

§ 21. FUNCTIONS OF 45°, 135°, ..., *i.e.*, OF $n\pi \pm \frac{\pi}{4}$.

THM. 13. *The functions of* $n\pi \pm \frac{\pi}{4}$ *are:*

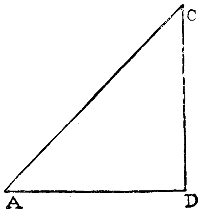
Angle.	Sine.	Cos.	Tan.	Cot.	Sec.	Csc.
$45°$, *i.e.*, $\frac{\pi}{4}$; or $2n\pi + \frac{\pi}{4}$	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$	$+1$	$+1$	$\sqrt{2}$	$\sqrt{2}$
$135°$, *i.e.*, $\frac{3\pi}{4}$; or $(2n+1)\pi - \frac{\pi}{4}$	$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$	-1	-1	$-\sqrt{2}$	$\sqrt{2}$
$225°$, *i.e.*, $\frac{5\pi}{4}$; or $(2n+1)\pi + \frac{\pi}{4}$	$-\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$	$+1$	$+1$	$-\sqrt{2}$	$-\sqrt{2}$
$315°$, *i.e.*, $\frac{7\pi}{4}$; or $2n\pi - \frac{\pi}{4}$	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$	-1	-1	$\sqrt{2}$	$-\sqrt{2}$

For, let ADC be a right isosceles triangle, right-angled at D; then $\angle A = 45°$,

and $AD = DC = AC \cdot \sqrt{\frac{1}{2}}$. [*geom.*

But $\sin A = DC : AC$, and $\cos A = AD : AC$.

$\therefore\ \sin 45° = \sqrt{\frac{1}{2}}$, and $\cos 45° = \sqrt{\frac{1}{2}}$.



The reader may prove the remaining values as exercises upon the preceding theorems; for example:

$$\tan 45° = \frac{\sin 45°}{\cos 45°} = \frac{\sqrt{\frac{1}{2}}}{\sqrt{\frac{1}{2}}} = 1;$$

$$\sec 45° = \frac{1}{\cos 45°} = \frac{1}{\sqrt{\frac{1}{2}}} = \sqrt{2};$$

$$\begin{aligned}
\tan\ 225° &= \tan(180° + 45°) = \tan 45° = 1; \\
\sin\ 405° &= \sin\ 45° = \sqrt{\tfrac{1}{2}}; \\
\cos -405° &= \cos -45° = \cos 45° = \sqrt{\tfrac{1}{2}}; \\
\tan\ 585° &= \tan\ 225° = \tan 45° = 1; \\
\cot -585° &= \cot\ 135° = -\tan 45° = -1; \\
\sec\ 675° &= \sec -45° = \sec 45° = \sqrt{2}; \\
\csc -675° &= \csc\ 45° = \sqrt{2}.
\end{aligned}$$

§ 22. FUNCTIONS OF 0°, 90°, 180°, ..., *i.e.*, OF $\frac{n}{2}\pi$.

THM. 14. *The functions of* $\frac{n}{2}\pi$ *are:*

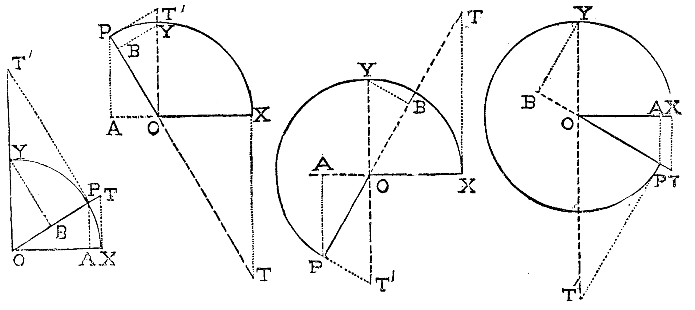
Angle.	Sine.	Cos.	Tan.	Cot.	Sec.	Csc.
0°, *i.e.*, 0π; or $2n\pi$	0	1	0	∞	1	∞
90°, *i.e.*, $\frac{\pi}{2}$; or $(2n+\frac{1}{2})\pi$	1	0	∞	0	∞	1
180°, *i.e.*, π; or $(2n+1)\pi$	0	-1	0	∞	-1	∞
270°, *i.e.*, $\frac{3\pi}{2}$; or $(2n-\frac{1}{2})\pi$	-1	0	∞	0	∞	-1

The reader may prove these values by direct reference to the definitions of the functions. [§ 7

NOTE. From the values given above, and from others computed by methods to be given later, tables have been constructed, giving the functions of all angles, and other tables giving the logarithms of these functions; they are called *trigonometric tables*. For their use the reader is referred to the tables themselves.

§ 23. GRAPHICAL REPRESENTATION OF FUNCTIONS.

Line-Functions.



Let XOP be any angle, and with O as center, and any radius OX, describe a circle, cutting the sides in X and P; from P let fall PA ⊥ OX; at X draw XT tangent to XP, and meeting OP in T. Draw OY ⊥ OX, and meeting the circle at Y; from Y let fall YB ⊥ OP, and at P draw PT′ tangent to PY, and meeting OY in T′; then: if the radius be taken as the unit of length, the ratio AP : OX is the numerical measure of AP; *i.e.*, the number of units in the length of the line AP is equal to sin XOP,

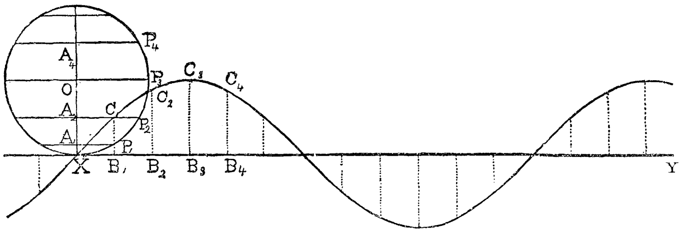
and the line AP *represents* sin XOP.
So, the line OA represents cos XOP;
the line XT represents tan XOP;
the line OT represents sec XOP;
the line AX represents vers XOP.
So, the line BY represents sin POY, *i.e.*, cos XOP;
the line PT′ represents tan POY, *i.e.*, cot XOP;
the line OT′ represents sec POY, *i.e.*, csc XOP;
the line BP represents vers POY, *i.e.*, covers XOP.

These lines may be called the *line-functions* of the angles, as distinguished from the *ratio-functions* heretofore defined. They are also called *functions of arcs;* and the others, *functions of angles.*

In most of the earlier treatises on trigonometry the functions were always defined as lines, thus: the sine was said to be "the perpendicular from one extremity of an arc upon the diameter through the other extremity"; the cosine was "the distance from the center to the foot of the sine," or "the sine of the complement," and so on.

NOTE 1. These line-functions must never be thought of as identical with the ratio-functions, not even when the radius is unity; the ratio-functions are neither lines nor the lengths of lines, but numbers merely whose values are independent of any measuring unit of length. The reader may name the line-functions in the different figures, may state which of them are positive and which negative, and may show that the ratios of these line-functions to the radius are identical with the ratio-functions heretofore given.

Curve of Sines.



Let OX be the radius of a circle, and divide the circumference into any convenient parts at P_1, P_2, ... ; draw A_1P_1, A_2P_2, ..., ordinates of P_1, P_2, ... with reference to OX, and sines of the arcs XP_1, XP_2, ... ; draw XY tangent to the circle at X.
Conceive the circle to roll along XY ; let B_1, B_2, ... be the points on XY where P_1, P_2, ... rest respectively ; and at B_1, B_2, ... erect perpendiculars to XY, and make $B_1C_1 = A_1P_1$, $B_2C_2 = A_2P_2$, ...
Through C_1, C_2, ... draw a smooth curve ; it is the *curve of sines.*

NOTE 2. The following relations are manifest :

(1) At X the sine is 0.

(2) While the arc is small, the sine is nearly as long as the arc.

(3) The sine increases more and more slowly.

(4) It is equal to +1, its maximum, when the arc is a quadrant ; then

(5) It decreases, at first very slowly, but faster and faster as the arc approaches a half-circle.

(6) When the arc is a half-circle, the sine is 0.

(7) While the arc increases from 180° to 270°, the sine decreases from 0 to −1, its minimum.

(8) While the arc increases from 270° to 360°, the sine increases from −1 to 0.

(9) At the end of the first revolution the sine is again 0.

(10) During every successive revolution the same phenomena are repeated in the same order, and for negative as well as positive arcs.

(11) While the revolution is continuous, the values of the sines are periodic, every successive revolution indicating a new cycle, and a new wave in the curve.

(12) The four parts of each wave that correspond to the four quadrants of the angle XOP, are equal and similar to each other.

(13) The sine has no value greater than $+1$, nor less than -1.

The reader may draw curves to represent the other functions, and discuss them; he will find, among other things, that:

(14) The tangent is 0 at $\theta = 0$;
increases through the first quadrant to $+\infty$;
at 90° changes suddenly to $-\infty$;
increases through the second quadrant to 0 at 180°;
increases through the third quadrant to $+\infty$ at 270°;
at 270° changes to $-\infty$;
increases through the fourth quadrant to 0 at 360°; and so on.

(15) The secant is $+1$ at $\theta = 0$;
increases through the first quadrant to $+\infty$;
at 90° changes to $-\infty$;
increases through the second quadrant to -1 at 180°;
decreases through the third quadrant to $-\infty$ at 270°;
at 270° changes to $+\infty$;
decreases through the fourth quadrant to $+1$ at 360°; and so on.
It has no value between $+1$ and -1.

(16) The cosine, cotangent and cosecant have the same limits as the sine, tangent and secant respectively; they go through like changes and are represented by like curves; but they begin, for $\theta = 0$, with different values, viz., 1, ∞, and ∞.

§ 24. EXERCISES.

1. Express in degree-measure the angles:

$$\frac{\pi}{5},\ \frac{\pi}{7},\ \frac{\pi}{9},\ \frac{5\pi}{3},\ 3.1416,\ -.7854,\ 1,\ 1.5,\ -2,\ \pi+1,\ \frac{3\pi+1}{6}.$$

2. Express in π-measure the angles:

$$14°,\ 15°,\ 24°,\ 120°,\ 137°\,15',\ -4800°,\ 13',\ 24'',\ -5°,\ 19'\,37''.5.$$

3. If the radius of a circle be one inch, what is the length of the arcs:

$$14°,\ 15°,\ 120°,\ 57°\,17'\,44''.8,\ 1°,\ 1',\ 1'',\ \frac{\pi}{5},\ \frac{\pi}{7},\ 2,\ \pi+1.$$

4. Find the complements and the supplements of the angles:

$37°,\ 215°,\ 325°,\ 107°\,12'\,15'',\ -36°\,12',\ \frac{\pi}{5},\ \frac{6\pi}{7},\ \frac{11\pi}{9},\ -\frac{5\pi}{3}.$

5. Construct the points in a plane whose abscissas and ordinates are:

$5,3;\ \ 2,9;\ \ 8,-7;\ \ -3,4;\ \ -5,-8.$ [Use any convenient unit.]

6. Construct the points in a plane whose bearings and distances are:

$30°,10;\ 30°,-10;\ 210°,10;\ -150°,10;\ 150°,-10;\ -60°,10;$
$\pi,8;\ \pi,-8;\ -\pi,8;\ 0°,8;\ \frac{5\pi}{3},4;\ \frac{5\pi}{3},-4;\ -\frac{\pi}{3},4;\ -\frac{\pi}{3},-4.$

7. Construct the angles whose

sines are $\frac{1}{2},\ -\frac{2}{3},\ \frac{\sqrt{3}-1}{2},\ 1,\ 0,\ \frac{\frac{1}{2}+\sqrt{3}}{6}.$

cosines are $.6,\ \pm\frac{1}{2},\ \pm\frac{1}{2}\sqrt{3},\ \frac{1-\sqrt{5}}{2}.$

tangents are $\frac{1}{3},\ \frac{3}{4},\ 0,\ -1,\ \infty,\ \sqrt{2}+1.$

cotangents are $\frac{1}{4},\ \frac{1}{20},\ \frac{2a-b}{b\sqrt{3}},$ [a and b, any lines.

8. Write formulae for all values of θ:

when $\sin\theta = -\sin\alpha,\ \sqrt{\sin^2\alpha},\ \sqrt{\cos^2\alpha}.$

when $\cos\theta = -\cos\alpha,\ \sqrt{\cos^2\alpha},\ \sqrt{\sin^2\alpha}.$

when $\tan\theta = -\tan\alpha,\ \sqrt{\tan^2\alpha},\ \sqrt{\cot^2\alpha}.$

when $\cot\theta = -\cot\alpha,\ \sqrt{\cot^2\alpha},\ \sqrt{\tan^2\alpha}.$

9. Find the remaining functions of θ:

if $\sin\theta = .6,\ \frac{8}{17},\ \frac{3}{4},\ -\frac{2}{3},\ \frac{1}{\sqrt{5}},\ \frac{1}{\sqrt{2}}+\frac{\sqrt{3}}{6}.$

if $\cos\theta = \frac{1}{2},\ -\frac{\sqrt{3}}{2},\ \frac{12}{13},\ \frac{1+\sqrt{3}}{2\sqrt{2}}.$

if $\tan\theta = \frac{3}{4},\ -\frac{1}{3},\ \sqrt{\frac{(s-b)(s-c)}{s(s-a)}},$ [$s = \frac{1}{2}(a+b+c)$.

if $\cot\theta = \frac{1}{4},\ \frac{2\sqrt{10}}{3},\ \frac{2a}{1-a^2},\ -\frac{a}{b}.$

if $\sec\theta = \frac{5}{4},\ \frac{85}{77},\ -\frac{65}{16},\ \frac{\sqrt{41}}{4},\ \frac{\sqrt{82}}{9}.$

if $\csc\theta = -2,\ \frac{5}{3},\ \frac{1+b^2}{2b},\ \frac{b}{a},\ 2(-1)^m.$ [m any integer.

*10. Find $\sin\theta$ from the equation, $\sin\theta \,.\, \cos\theta = m$.

*11. If $\tan\theta + \cot\theta = m$, express all the functions of θ in terms of m.

*12. If $\tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{2}}$, find the value of $\frac{a}{\cos\theta} + \frac{b}{\sin\theta}$.

*13. Eliminate θ from the equations:

$$\sin\theta + \cos\theta = a, \quad \tan\theta + \sec\theta = 2b.$$

*14. Eliminate ϕ from the equations:

$$\csc\phi - \sin\phi = m, \quad \sec\phi - \cos\phi = n.$$

*15. Eliminate θ and ϕ from the equations:
$a\sin^2\theta + b\cos^2\theta = m, \quad b\sin^2\phi + a\cos^2\phi = n, \quad a\tan\theta = b\tan\phi$.

*16. If $\tan\phi + \sec\phi = a$, find $\sin\phi$.

*17. Show geometrically that $\sin 2\phi < 2\sin\phi$, if $\phi > 0, < \pi$.

*18. Show how to divide an angle into two parts which shall have their sines in a given ratio; their cosines in a given ratio.

*19. Show how to construct an angle whose tangent is four times its sine.

*20. Find all the angles whose

sines are	$\pm\sqrt{\frac{1}{2}}$,	$\pm\frac{1}{2}\sqrt{3}$,	$\sin 50°$,	$\cos 50°$,	$\cos\frac{\pi}{16}$.
cosines are	$\pm\sqrt{\frac{1}{2}}$,	$\pm\frac{1}{2}\sqrt{3}$,	$\cos 50°$,	$\sin 50°$,	$\sin\frac{\pi}{16}$.
tangents are	± 1,	$\pm\sqrt{3}$,	$\tan\frac{\pi}{8}$,	$\cot 40°$,	$\cot\frac{\pi}{8}$.
cotangents are	0,	$\pm\sqrt{3}$,	$\cot\frac{3\pi}{8}$,	$\tan 20°$,	$\tan\frac{3\pi}{8}$.
secants are	∞,	$\pm\frac{2}{\sqrt{3}}$,	$\sec\frac{5\pi}{8}$,	$\csc 35°$,	$\csc\frac{5\pi}{8}$.
cosecants are	∞,	± 2,	$\csc\frac{7\pi}{8}$,	$\sec 35°$,	$\sec\frac{7\pi}{8}$.

21. Find the
sine of $225^\circ, \quad -585^\circ, \quad 810^\circ, \quad -960^\circ, \quad 3\pi, \quad -\frac{27\pi}{4}.$
cosine of $315^\circ, \quad -675^\circ, \quad 960^\circ, \quad -1110^\circ, \quad \frac{23\pi}{6}, \quad -8\pi.$
tangent of $495^\circ, \quad -945^\circ, \quad 1110^\circ, \quad -1260^\circ, \quad \frac{28\pi}{6}, \quad -\frac{37\pi}{4}.$
cotangent of $675^\circ, \quad -1035^\circ, \quad 1260^\circ, \quad -1410^\circ, \quad \frac{33\pi}{6}, \quad -\frac{42\pi}{4}.$
secant of $855^\circ, \quad -1215^\circ, \quad 1410^\circ, \quad -1560^\circ, \quad \frac{38\pi}{6}, \quad -\frac{47\pi}{4}.$
cosecant of $1035^\circ, \quad -1395^\circ, \quad 1560^\circ, \quad -1710^\circ, \quad \frac{43\pi}{6}, \quad -13\pi.$

22. In terms of the functions of positive angles less than 90°, express the values of the
sine of $135^\circ, \quad 335^\circ, \quad -535^\circ, \quad -735^\circ, \quad \frac{27\pi}{5}, \quad -\frac{29\pi}{7}.$
cosine of $235^\circ, \quad 435^\circ, \quad -635^\circ, \quad -835^\circ, \quad \frac{29\pi}{5}, \quad -\frac{31\pi}{7}.$
tangent of $335^\circ, \quad 535^\circ, \quad -735^\circ, \quad -935^\circ, \quad \frac{31\pi}{5}, \quad -\frac{33\pi}{7}.$
cotangent of $435^\circ, \quad 635^\circ, \quad -835^\circ, \quad -1035^\circ, \quad \frac{33\pi}{5}, \quad -5\pi.$
secant of $535^\circ, \quad 735^\circ, \quad -935^\circ, \quad -1135^\circ, \quad 7\pi, \quad -\frac{37\pi}{7}.$
cosecant of $635^\circ, \quad 835^\circ, \quad -1035^\circ, \quad -1235^\circ, \quad \frac{37\pi}{5}, \quad -\frac{39\pi}{7}.$

23. In terms of the functions of positive angles less than 45°, express the values of the
sine of $50^\circ, \quad 150^\circ, \quad -250^\circ, \quad -350^\circ, \quad \frac{3\pi}{12}, \quad -4\pi.$
cosine of $60^\circ, \quad 160^\circ, \quad -260^\circ, \quad -360^\circ, \quad \frac{5\pi}{12}, \quad -\frac{14\pi}{3}.$
tangent of $70^\circ, \quad 170^\circ, \quad -270^\circ, \quad -370^\circ, \quad \frac{7\pi}{12}, \quad -\frac{16\pi}{3}.$
cotangent of $80^\circ, \quad 180^\circ, \quad -280^\circ, \quad -380^\circ, \quad \frac{9\pi}{12}, \quad -6\pi.$

secant of $90^\circ,\ 190^\circ,\ -290^\circ,\ -390^\circ,\ \frac{11\pi}{12},\ -\frac{20\pi}{3}$.
cosecant of $100^\circ,\ 200^\circ,\ -300^\circ,\ -400^\circ,\ \frac{13\pi}{12},\ -\frac{22\pi}{3}$.

*24. Trace the changes, when θ increases from 0 to 2π, in the sign and value of each of the expressions:

$\sin\theta+\cos\theta,\quad \sin\theta-\cos\theta,\quad \sin\theta+\csc\theta,\quad \tan\theta-\cot\theta,$
$\sin^2\theta,\quad \cos^2\theta,\quad \sin^2\theta-\cos^2\theta,\quad \cos^2\theta-\sin^2\theta,\quad \tan^2\theta+\cot^2\theta.$

25. From the table of natural functions, find the
sin of 20°, 21°, 20° 10′, 20° 10′ 45″, 89° 18′ 25″, 157° 15′ 23″.
cos of 20°, 21°, 20° 10′, 20° 10′ 45″, 89° 18′ 25″, 157° 15′ 23″.
tan of 35°, 36°, 35° 15′, 35° 15′ 47″, 89° 58′ 35″, 125° 0′ 12″.
cot of 35°, 36°, 35° 15′, 35° 15′ 47″, 89° 58′ 35″, 125° 0′ 12″.

26. From the table of natural functions find the angles whose
sines are .25882, .25910, .25900, .92794, .92805, .92800.
cosines are .92794, .92805, .92800, .25910, .25882, .25900.
tangents are .5022, .5059, .5035, .9217, .9271, .9250.
cotangents are .9217, .9271, .9250, .5022, .5059, .5035.

27. From the table of logarithmic functions find the logarithmic
sin of 20°, 21°, 20° 10′, 20° 10′ 45″, 89° 18′ 25″, 157° 15′ 23″.
cos of 20°, 21°, 20° 10′, 20° 10′ 45″, 89° 18′ 25″, 157° 15′ 23″.
tan of 35°, 36°, 35° 15′, 35° 15′ 47″, 89° 58′ 35″, 125° 0′ 12″.
cot of 35°, 36°, 35° 15′, 35° 15′ 47″, 89° 58′ 35″, 125° 0′ 12″.
sec of 50°, 51°, 50° 20′, 50° 20′ 49″, 115° 0′ 45″, 179° 58′ 55″.
csc of 50°, 51°, 50° 20′, 50° 20′ 49″, 115° 0′ 45″, 179° 58′ 55″.

28. From the tables of logarithmic functions, find the angles whose logarithmic
sines are 8.580892, 8.584193, 8.582125, 9.999683.
cosines are 8.580892, 8.584193, 8.582125, 9.999683.
tangents are 8.581208, 8.584514, 8.583125, 11.418790.
cotangents are 8.581208, 8.584514, 8.583125, 11.418790.
secants are 10.367529, 11.367514, 12.367529, 13.367514.
cosecants are 10.367529, 11.367514, 12.367529, 13.367514.

II. GENERAL FORMULAE.

§ 1. FUNCTIONS OF THE SUM, AND OF THE DIFFERENCE, OF TWO ANGLES.

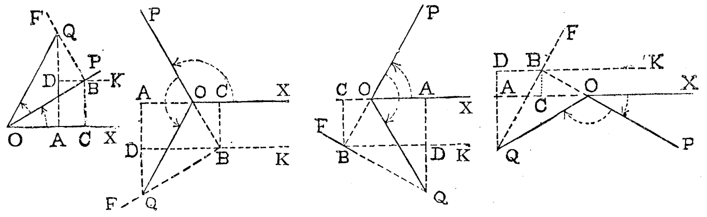
THEOREM 1. *If* θ *and* θ' *be any two angles, then:*

39] $\sin(\theta+\theta') = \sin\theta\cos\theta' + \cos\theta\sin\theta'$;

40] $\sin(\theta-\theta') = \sin\theta\cos\theta' - \cos\theta\sin\theta'$;

41] $\cos(\theta+\theta') = \cos\theta\cos\theta' - \sin\theta\sin\theta'$;

42] $\cos(\theta-\theta') = \cos\theta\cos\theta' + \sin\theta\sin\theta'$.



For, let XOP be any angle θ, positive or negative, and POQ be any other angle θ'; then $XOQ = \theta + \theta'$.
Draw BQ ordinate of Q with reference to OP, and AQ and CB ordinates of Q and B with reference to OX; draw BK parallel to OX and meeting AQ in D; on BQ take F so that $\angle KBF = \frac{1}{2}\pi + \theta$. Then:

(1) $\sin(\theta+\theta') = \dfrac{\text{ordinate AQ}}{\text{distance OQ}} = \dfrac{CB + DQ}{OQ}$; [I. § 7

but $\dfrac{CB}{OQ} = \dfrac{CB}{OB} \cdot \dfrac{OB}{OQ}$

$= \sin\theta\cos\theta'$, [I. § 7

and $\dfrac{DQ}{OQ} = \dfrac{DQ}{BQ} \cdot \dfrac{BQ}{OQ}$

$= \sin(\frac{1}{2}\pi + \theta)\sin\theta'$ [I. § 7

$= \cos\theta\sin\theta'$; [7

$\therefore$ $\sin(\theta+\theta') = \sin\theta\cos\theta' + \cos\theta\sin\theta'$. Q. E. D.

(2) In [39] substitute $-\theta'$ for θ', then:

$$\because \quad \theta-\theta'=\theta+(-\theta'),$$

$$\therefore \quad \sin(\theta-\theta')=\sin\theta\cos(-\theta')+\cos\theta\sin(-\theta') \qquad [39$$

$$=\sin\theta\cos\theta'-\cos\theta\sin\theta'. \qquad \text{Q.E.D.} \quad [1, 2$$

(3) $$\cos(\theta+\theta')=\frac{\text{abscissa OA}}{\text{distance OQ}}=\frac{\text{OC}+\text{BD}}{\text{OQ}}; \qquad [\text{I. § 7}$$

but $$\frac{\text{OC}}{\text{OQ}}=\frac{\text{OC}}{\text{OB}}\cdot\frac{\text{OB}}{\text{OQ}}$$

$$=\cos\theta\cos\theta', \qquad [\text{I. § 7}$$

and $$\frac{\text{BD}}{\text{OQ}}=\frac{\text{BD}}{\text{BQ}}\cdot\frac{\text{BQ}}{\text{OQ}}$$

$$=\cos(\tfrac{1}{2}\pi+\theta)\sin\theta' \qquad [\text{I. § 7}$$

$$=-\sin\theta\sin\theta'; \qquad [8$$

$$\therefore \quad \cos(\theta+\theta')=\cos\theta\cos\theta'-\sin\theta\sin\theta'. \qquad \text{Q.E.D.}$$

(4) In [41] substitute $-\theta'$ for θ', then

$$\cos(\theta-\theta')=\cos\theta\cos(-\theta')-\sin\theta\sin(-\theta') \qquad [41$$

$$=\cos\theta\cos\theta'+\sin\theta\sin\theta'. \qquad \text{Q.E.D.} \quad [2, 1$$

NOTE. Since each of the angles θ and θ' may be in either of the four quadrants, there are sixteen different cases possible; but the proof is general, for it applies to all alike. The reader may enumerate the cases in detail and draw the other twelve figures.

COR. 1. *If θ and θ' be any two angles, then:*

43] $$\tan(\theta+\theta')=\frac{\tan\theta+\tan\theta'}{1-\tan\theta\tan\theta'};$$

44] $$\tan(\theta-\theta')=\frac{\tan\theta-\tan\theta'}{1+\tan\theta\tan\theta'}.$$

For $$\tan(\theta+\theta')=\frac{\sin(\theta+\theta')}{\cos(\theta+\theta')} \qquad [35$$

$$=\frac{\sin\theta\cos\theta'+\cos\theta\sin\theta'}{\cos\theta\cos\theta'-\sin\theta\sin\theta'}. \qquad [39, 41$$

Divide both terms of the fraction by $\cos\theta\cos\theta'$, then

$$\tan(\theta+\theta')=\frac{\tan\theta+\tan\theta'}{1-\tan\theta\tan\theta'}. \qquad \text{Q.E.D.} \quad [35$$

So, $$\tan(\theta-\theta')=\frac{\sin(\theta-\theta')}{\cos(\theta-\theta')} \qquad [35$$

$$=\frac{\sin\theta\cos\theta'-\cos\theta\sin\theta'}{\cos\theta\cos\theta'+\sin\theta\sin\theta'}$$

$$=\frac{\tan\theta-\tan\theta'}{1+\tan\theta\tan\theta'}. \qquad \text{Q. E. D.}$$

The reader may find like formulae for $\cot(\theta+\theta')$ and $\cot(\theta-\theta')$.

NOTE. The six formulae [39–44] may be written as three:

39, 40] $\sin(\theta\pm\theta')=\sin\theta\cos\theta'\pm\cos\theta\sin\theta'$;

41, 42] $\cos(\theta\pm\theta')=\cos\theta\cos\theta'\mp\sin\theta\sin\theta'$;

43, 44] $\tan(\theta\pm\theta')=\dfrac{\tan\theta\pm\tan\theta'}{1\mp\tan\theta\tan\theta'}$.

COR. 2. *If θ and θ' be any two angles, then:*

45] $\dfrac{\sin(\theta\pm\theta')}{\sin\theta\sin\theta'}=\cot\theta'\pm\cot\theta$;

46] $\dfrac{\cos(\theta\pm\theta')}{\sin\theta\sin\theta'}=\cot\theta\cot\theta'\mp 1$;

47] $\dfrac{\sin(\theta\pm\theta')}{\cos\theta\cos\theta'}=\tan\theta\pm\tan\theta'$;

48] $\dfrac{\cos(\theta\pm\theta')}{\cos\theta\cos\theta'}=1\mp\tan\theta\tan\theta'$;

49] $\dfrac{\sin(\theta\pm\theta')}{\sin\theta\cos\theta'}=1\pm\cot\theta\tan\theta'$;

50] $\dfrac{\cos(\theta\pm\theta')}{\sin\theta\cos\theta'}=\cot\theta\mp\tan\theta'$.

The reader may prove these formulae by performing the divisions and reductions indicated.

COR. 3. *If θ and θ' be any two angles, then:*

51] $\sin(\theta+\theta')+\sin(\theta-\theta')=2\sin\theta\cos\theta'$;

52] $\sin(\theta+\theta')-\sin(\theta-\theta')=2\cos\theta\sin\theta'$;

53] $\cos(\theta+\theta')+\cos(\theta-\theta')=2\cos\theta\cos\theta'$;

54] $\cos(\theta+\theta')-\cos(\theta-\theta')=-2\sin\theta\sin\theta'$.

For, if to [39], [40] be added, the result is [51];
if from [39], [40] be subtracted, the result is [52];
if to [41], [42] be added, the result is [53];
if from [41], [42] be subtracted, the result is [54].

COR. 4. *If θ and θ' be any two angles, then:*

55] $\sin(\theta+\theta')\sin(\theta-\theta') = \sin^2\theta - \sin^2\theta' = \cos^2\theta' - \cos^2\theta;$

56] $\cos(\theta+\theta')\cos(\theta-\theta') = \cos^2\theta - \sin^2\theta' = \cos^2\theta' - \sin^2\theta.$

The reader may prove, by performing the multiplications indicated; he will make use of [36].

COR. 5. *If θ and θ' be any two angles, then:*

57] $\dfrac{\sin(\theta+\theta')}{\sin(\theta-\theta')} = \dfrac{\tan\theta + \tan\theta'}{\tan\theta - \tan\theta'};$

58] $\dfrac{\cos(\theta+\theta')}{\cos(\theta-\theta')} = \dfrac{1 - \tan\theta\tan\theta'}{1 + \tan\theta\tan\theta'};$

59] $\dfrac{\sin(\theta\pm\theta')}{\cos(\theta\mp\theta')} = \dfrac{\tan\theta \pm \tan\theta'}{1 \pm \tan\theta\tan\theta'}.$

The reader may prove, by aid of [47–48].

§ 2. FUNCTIONS OF DOUBLE ANGLES, AND OF HALF ANGLES.

THM. 2. *If θ be any angle, then:*

60] $\sin 2\theta = 2\sin\theta\cos\theta;$

61] $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta;$

62] $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta};$

63] $\sin\frac{1}{2}\theta = \sqrt{\dfrac{1 - \cos\theta}{2}};$

64] $\cos\frac{1}{2}\theta = \sqrt{\dfrac{1 + \cos\theta}{2}};$

65] $\tan\frac{1}{2}\theta = \sqrt{\dfrac{1 - \cos\theta}{1 + \cos\theta}} = \dfrac{\sin\theta}{1 + \cos\theta} = \dfrac{1 - \cos\theta}{\sin\theta}.$

(1) In [39, 41, 43] substitute θ for θ', then:

$\because\quad \theta + \theta' = 2\theta;$

$\therefore\quad \sin 2\theta = \sin\theta\cos\theta + \cos\theta\sin\theta = 2\sin\theta\cos\theta.$ [39

So, $\quad \cos 2\theta = \cos\theta\cos\theta - \sin\theta\sin\theta = \cos^2\theta - \sin^2\theta$ [41

$\qquad = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta;$ [36

and $\quad \tan 2\theta = \dfrac{\tan\theta + \tan\theta}{1 - \tan\theta\tan\theta} = \dfrac{2\tan\theta}{1 - \tan^2\theta}.$ Q. E. D. [43

(2) $\because\quad \cos 2\theta = 1 - 2\sin^2\theta,$ whatever the angle θ, [61

$\therefore\quad \cos\theta = 1 - 2\sin^2\frac{1}{2}\theta,$ [substitute $\frac{1}{2}\theta$ for θ

$\therefore\quad \sin\frac{1}{2}\theta = \sqrt{\dfrac{1 - \cos\theta}{2}}.$

So, $\because\quad \cos 2\theta = 2\cos^2\theta - 1,$ whatever the angle θ, [61

$\therefore\quad \cos\theta = 2\cos^2\frac{1}{2}\theta - 1,$ [substitute $\frac{1}{2}\theta$ for θ

$\therefore\quad \cos\frac{1}{2}\theta = \sqrt{\dfrac{1 + \cos\theta}{2}}.$

Hence, $\quad \tan\frac{1}{2}\theta = \dfrac{\sin\frac{1}{2}\theta}{\cos\frac{1}{2}\theta} = \sqrt{\dfrac{1 - \cos\theta}{1 + \cos\theta}}.$ [35

Multiply both terms of the fraction by $\sqrt{(1 + \cos\theta)}$, then

$\tan\frac{1}{2}\theta = \dfrac{\sqrt{(1 - \cos^2\theta)}}{1 + \cos\theta} = \dfrac{\sin\theta}{1 + \cos\theta};$ Q. E. D. [36

or, multiply both terms of the fraction by $\sqrt{(1 - \cos\theta)}$, then

$\tan\frac{1}{2}\theta = \dfrac{1 - \cos\theta}{\sqrt{(1 - \cos^2\theta)}} = \dfrac{1 - \cos\theta}{\sin\theta}.$ Q. E. D. [36

NOTE. Although the radical $\sqrt{(1 - \cos^2\theta)}$, $= \pm\sin\theta$, has two values, opposites; yet $\tan\frac{1}{2}\theta$ has but one sign and one value; for, when θ is in the first or second quadrant, then $\frac{1}{2}\theta$ is in the first or third quadrant, and $\sin\theta$ and $\tan\frac{1}{2}\theta$ are both positive; so, when θ is in the third or fourth quadrant, then $\frac{1}{2}\theta$ is in the second or fourth quadrant, and $\sin\theta$ and $\tan\frac{1}{2}\theta$ are both negative; *i.e.*, whatever the sign of $\sin\theta$, the same is that of $\tan\frac{1}{2}\theta$. [I. Th. 3

COR. *If θ be any angle, then:*

66] $\quad \sin\frac{1}{2}\theta = \frac{1}{2}[\sqrt{(1 + \sin\theta)} - \sqrt{(1 - \sin\theta)}];$

67] $\quad \cos\frac{1}{2}\theta = \frac{1}{2}[\sqrt{(1 + \sin\theta)} + \sqrt{(1 - \sin\theta)}].$ [See nt. 1 p. 103

For, ∵ $\sin 2\theta = 2\sin\theta\cos\theta$, whatever the angle θ, [60
∴ $\sin\theta = 2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta$; [substitute $\frac{1}{2}\theta$ for θ
but $1 = \cos^2\frac{1}{2}\theta + \sin^2\frac{1}{2}\theta$; [36
∴ $1+\sin\theta = (\cos\frac{1}{2}\theta + \sin\frac{1}{2}\theta)^2$,
and $1-\sin\theta = (\cos\frac{1}{2}\theta - \sin\frac{1}{2}\theta)^2$;
∴ $\sqrt{(1+\sin\theta)} = \cos\frac{1}{2}\theta + \sin\frac{1}{2}\theta$,
and $\sqrt{(1-\sin\theta)} = \cos\frac{1}{2}\theta - \sin\frac{1}{2}\theta$.
∴ etc. Q. E. D.

§ 3. FUNCTIONS OF THE HALF-SUM, AND OF THE HALF-DIFFERENCE, OF TWO ANGLES.

Thm. 3. *If θ and θ' be any two angles, then:*

68] $\sin\theta + \sin\theta' = 2\sin\frac{1}{2}(\theta+\theta')\cos\frac{1}{2}(\theta-\theta')$;
69] $\sin\theta - \sin\theta' = 2\cos\frac{1}{2}(\theta+\theta')\sin\frac{1}{2}(\theta-\theta')$;
70] $\cos\theta + \cos\theta' = 2\cos\frac{1}{2}(\theta+\theta')\cos\frac{1}{2}(\theta-\theta')$;
71] $\cos\theta - \cos\theta' = -2\sin\frac{1}{2}(\theta+\theta')\sin\frac{1}{2}(\theta-\theta')$.

For, in [51–54] substitute $\frac{1}{2}(\theta+\theta')$ for θ and $\frac{1}{2}(\theta-\theta')$ for θ';
then also θ stands for $(\theta+\theta')$, and θ' for $(\theta-\theta')$;
and [51–54] reduce to [68–71] respectively.

Cor. *If θ and θ' be any two angles:*

72] $\dfrac{\sin\theta+\sin\theta'}{\sin\theta-\sin\theta'} = \dfrac{\tan\frac{1}{2}(\theta+\theta')}{\tan\frac{1}{2}(\theta-\theta')}$;

73] $\dfrac{\sin\theta+\sin\theta'}{\cos\theta+\cos\theta'} = \tan\frac{1}{2}(\theta+\theta')$;

74] $\dfrac{\sin\theta-\sin\theta'}{\cos\theta+\cos\theta'} = \tan\frac{1}{2}(\theta-\theta')$;

75] $\dfrac{\sin\theta+\sin\theta'}{\cos\theta-\cos\theta'} = -\cot\frac{1}{2}(\theta-\theta')$;

76] $\dfrac{\sin\theta-\sin\theta'}{\cos\theta-\cos\theta'} = -\cot\frac{1}{2}(\theta+\theta')$;

77] $\dfrac{\cos\theta+\cos\theta'}{\cos\theta-\cos\theta'} = -\cot\frac{1}{2}(\theta+\theta')\cot\frac{1}{2}(\theta-\theta')$.

The reader may prove, by dividing [68–71] one by another, as indicated, and reducing the quotients.

*§ 4. FUNCTIONS OF THE SUM OF THREE OR MORE ANGLES, OF TRIPLE ANGLES, ETC.

THM. 4. *If* θ, θ', θ'', ... *be any angles, then:*

78] $$\sin(\theta+\theta'+\theta'') = \left\{\begin{array}{l} \quad\sin\theta\cos\theta'\cos\theta'' + \cos\theta\sin\theta'\cos\theta'' \\ + \cos\theta\cos\theta'\sin\theta'' - \sin\theta\sin\theta'\sin\theta'' \end{array}\right.$$
$$= \cos\theta\cos\theta'\cos\theta''(\tan\theta + \tan\theta' + \tan\theta'' - \tan\theta\tan\theta'\tan\theta'');$$

79] $$\cos(\theta+\theta'+\theta'') = \left\{\begin{array}{l} \quad\cos\theta\cos\theta'\cos\theta'' - \cos\theta\sin\theta'\sin\theta'' \\ - \sin\theta\cos\theta'\sin\theta'' - \sin\theta\sin\theta'\cos\theta'' \end{array}\right.$$
$$= \cos\theta\cos\theta'\cos\theta''(1 - \tan\theta'\tan\theta'' - \tan\theta\tan\theta'' - \tan\theta\tan\theta');$$

80] $$\tan(\theta+\theta'+\theta'') = \frac{\tan\theta + \tan\theta' + \tan\theta'' - \tan\theta\tan\theta'\tan\theta''}{1 - \tan\theta'\tan\theta'' - \tan\theta\tan\theta'' - \tan\theta\tan\theta'}.$$

The reader may prove, by developing $\sin[\theta + (\theta' + \theta'')]$, $\cos[\theta + (\theta' + \theta'')]$, and $\tan[\theta + (\theta' + \theta'')]$ as functions of the sum of two angles θ and $(\theta' + \theta'')$, and then developing the functions of $(\theta' + \theta'')$ which are involved. He may also prove the result for $\tan(\theta + \theta' + \theta'')$ by dividing that for $\sin(\theta + \theta' + \theta'')$ by that for $\cos(\theta + \theta' + \theta'')$.

The reader may in like manner get the sine, cosine, and tangent of the sum of four angles, of five angles, and so on. For sine and cosine, he will find that:

81] $$\frac{\sin(\theta+\theta'+\theta''+\cdots\cdots)}{\cos\theta\cos\theta'\cos\theta''\cdots\cdots} = \left\{\begin{array}{l} \quad\Sigma\tan\theta - \Sigma\tan\theta\tan\theta'\tan\theta'' \\ + \Sigma\tan\theta\tan\theta'\tan\theta''\tan\theta'''\tan\theta^{\text{iv}} \\ - \cdots\cdots; \end{array}\right.$$

82] $$\frac{\cos(\theta+\theta'+\theta''+\cdots\cdots)}{\cos\theta\cos\theta'\cos\theta''\cdots\cdots} = \left\{\begin{array}{l} \quad 1 - \Sigma\tan\theta\tan\theta' \\ + \Sigma\tan\theta\tan\theta'\tan\theta''\tan\theta''' - \cdots\cdots; \end{array}\right.$$

wherein $\Sigma\tan\theta$ stands for the sum of the tangents of θ, θ', ..., $\Sigma\tan\theta\tan\theta'$ stands for the sum of the products of those tangents taken two and two; and so on.

By making $\theta = \theta' = \theta'' = \cdots\cdots$, he will find that:

83] $$\sin 3\theta = 3\sin\theta\cos^2\theta - \sin^3\theta$$
$$= 3\sin\theta - 4\sin^3\theta;$$ [36

84] $$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$
$$= -3\cos\theta + 4\cos^3\theta;$$

85] $$\sin 4\theta = 4\sin\theta\cos^3\theta - 4\sin^3\theta\cos\theta$$
$$= 4\sin\theta\cos\theta - 8\sin^3\theta\cos\theta;$$ [36

86] $$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$
$$= 1 - 8\cos^2\theta + 8\cos^4\theta;$$

87] $$\sin 5\theta = 5\sin\theta\cos^4\theta - 10\sin^3\theta\cos^2\theta + \sin^5\theta$$
$$= 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta;$$

88] $$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$
$$= 5\cos\theta - 20\cos^3\theta + 16\cos^5\theta;$$

and, in general, that:

89] $$\sin n\theta = c_{n,1}\sin\theta\cos^{n-1}\theta - c_{n,3}\sin^3\theta\cos^{n-3}\theta + c_{n,5}\sin^5\theta\cos^{n-5}\theta - \dots\dots;$$

90] $$\cos n\theta = \cos^n\theta - c_{n,2}\sin^2\theta\cos^{n-2}\theta + c_{n,4}\sin^4\theta\cos^{n-4}\theta - \dots\dots;$$

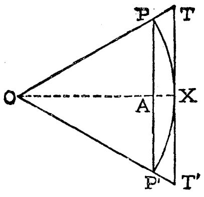
wherein $c_{n,r}$ denotes

$$\frac{n(n-1)(n-2)\dots\dots(n-r+1)}{1.2.3\dots\dots r},$$

the number of combinations of n things taken r at a time.

*§ 5. DIFFERENTIATION OF TRIGONOMETRIC FUNCTIONS.

LEMMA. *If θ be an infinitesimal angle, then* $\lim(\sin\theta : \theta) = 1$, *and* $\lim(\tan\theta : \theta) = 1$.



For, let c be the circumference of a circle, O the centre, p and p' the perimeters of two regular polygons of the same number of sides, the first inscribed in and the other circumscribed about the circle, and having their sides PP′, TT′, ... parallel each to each. Draw OAX ⊥ PP′ and TT′, then:

∵ $p < c < p'$, whatever the number of sides,

and ∵ p approaches indefinitely near to p' when the number of sides is indefinitely increased, [*geom.*

∴ c is the common limit of p and p',

∴ 1 is the common limit of the ratios $p : c$ and $p' : c$.

And $\because$ the two polygons are similar, and AP, XP and XT are like parts of p, c and p', [*geom.*

$\therefore\quad \text{AP}:\text{XP} = p:c,\quad$ and $\text{XT}:\text{XP} = p':c;$

$\therefore\quad \sin\theta:\theta, = \dfrac{\text{AP}}{\text{OP}}:\dfrac{\text{XP}}{\text{OP}} = \text{AP}:\text{XP}, = p:c,$

and $\quad \tan\theta:\theta, = \dfrac{\text{XT}}{\text{OX}}:\dfrac{\text{XP}}{\text{OX}} = \text{XT}:\text{XP}, = p':c;$

$\therefore\quad \lim(\sin\theta:\theta) = \lim(p:c) = 1,$

and $\quad \lim(\tan\theta:\theta) = \lim(p':c) = 1.$

THM. 5. *If* θ *be any angle, then:*

91] $\quad \text{D}_\theta \sin\theta = \cos\theta;\qquad \text{D}_\theta \csc\theta = -\cot\theta\csc\theta;$

92] $\quad \text{D}_\theta \cos\theta = -\sin\theta;\qquad \text{D}_\theta \sec\theta = \tan\theta\sec\theta;$

93] $\quad \text{D}_\theta \tan\theta = \sec^2\theta;\qquad \text{D}_\theta \cot\theta = -\csc^2\theta.$

For, let θ be any angle, and θ' an infinitesimal angle, the increment of θ, then:

$\because\quad \sin(\theta+\theta') - \sin\theta = 2\cos(\theta+\tfrac{1}{2}\theta')\sin\tfrac{1}{2}\theta';$ [69

$\therefore\quad \dfrac{\sin(\theta+\theta')-\sin\theta}{\theta'} = \cos(\theta+\tfrac{1}{2}\theta')\cdot\dfrac{\sin\frac{1}{2}\theta'}{\frac{1}{2}\theta'}.$

But $\quad \theta'$ is the increment of θ, [hypoth.

and $\quad \sin(\theta+\theta') - \sin\theta$ is the corresponding increment of $\sin\theta$.

$\therefore\quad \lim\dfrac{\text{inc}\sin\theta}{\text{inc}\,\theta}, = \text{D}_\theta\sin\theta, = \cos\theta.$ [*lemma*

So, $\because\quad \cos(\theta+\theta') - \cos\theta = -2\sin(\theta+\tfrac{1}{2}\theta')\sin\tfrac{1}{2}\theta';$ [71

$\therefore\quad \dfrac{\cos(\theta+\theta')-\cos\theta}{\theta'} = -\sin(\theta+\tfrac{1}{2}\theta')\dfrac{\sin\frac{1}{2}\theta'}{\frac{1}{2}\theta'};$

$\therefore\quad \lim\dfrac{\text{inc}\cos\theta}{\text{inc}\,\theta}, = \text{D}_\theta\cos\theta, = -\sin\theta.$ [*lemma*

So, $\quad \text{D}_\theta\tan\theta = \text{D}_\theta\dfrac{\sin\theta}{\cos\theta} = \dfrac{\cos\theta\,\text{D}_\theta\sin\theta - \sin\theta\,\text{D}_\theta\cos\theta}{\cos^2\theta}$

$\qquad = \dfrac{\cos^2\theta+\sin^2\theta}{\cos^2\theta} = \sec^2\theta.$ [36, 34

The reader may prove for $\text{D}_\theta\csc\theta, = \text{D}_\theta\dfrac{1}{\sin\theta}$, for $\text{D}_\theta\sec\theta$ and for $\text{D}_\theta\cot\theta$.

*§ 6. DEVELOPMENT OF TRIGONOMETRIC FUNCTIONS.

THM. 6. *If θ be any angle, then:*

94] $$\sin\theta = \theta - \frac{\theta^3}{3\,!} + \frac{\theta^5}{5\,!} - \frac{\theta^7}{7\,!} + \cdots\cdots;$$

95] $$\cos\theta = 1 - \frac{\theta^2}{2\,!} + \frac{\theta^4}{4\,!} - \frac{\theta^6}{6\,!} + \cdots\cdots;$$

For assume

$$\sin\theta = \mathrm{A} + \mathrm{B}\theta + \mathrm{C}\theta^2 + \mathrm{D}\theta^3 + \mathrm{E}\theta^4 + \mathrm{F}\theta^5 + \cdots\cdots,$$

wherein A, B, C, ... are unknown, but constant, whatever the value of θ; and differentiate both members of the equation, then

$$\cos\theta = \mathrm{B} + 2\mathrm{C}\theta + 3\mathrm{D}\theta^2 + 4\mathrm{E}\theta^3 + 5\mathrm{F}\theta^4 + \cdots\cdots. \qquad [91$$

So, $$-\sin\theta = 2\mathrm{C} + 6\mathrm{D}\theta + 12\mathrm{E}\theta^2 + 20\mathrm{F}\theta^3 + \cdots\cdots; \qquad [92$$

$$-\cos\theta = 6\mathrm{D} + 24\mathrm{E}\theta + 60\mathrm{F}\theta^2 + \cdots\cdots;$$

$$\sin\theta = 24\mathrm{E} + 120\mathrm{F}\theta + \cdots\cdots;$$

$$\cos\theta = 120\mathrm{F} + \cdots\cdots.$$

Let $\theta = 0$, then:

$$\sin\theta = \mathrm{A}; \qquad \therefore \mathrm{A} = 0;$$

$$\cos\theta = \mathrm{B}; \qquad \therefore \mathrm{B} = 1;$$

$$-\sin\theta = 2\mathrm{C}; \qquad \therefore \mathrm{C} = 0;$$

$$-\cos\theta = 6\mathrm{D} = 3\,!\,\mathrm{D}; \qquad \therefore \mathrm{D} = -\frac{1}{3\,!};$$

$$\sin\theta = 24\mathrm{E} = 4\,!\,\mathrm{E}; \qquad \therefore \mathrm{E} = 0;$$

$$\cos\theta = 120\mathrm{F} = 5\,!\,\mathrm{F}; \qquad \therefore \mathrm{F} = \frac{1}{5\,!};$$

and so on.

Substitute their values for A, B, C, ... in the assumed development, then

$$\sin\theta = \theta - \frac{\theta^3}{3\,!} + \frac{\theta^5}{5\,!} - \frac{\theta^7}{7\,!} + \cdots\cdots. \qquad \text{Q. E. D.}$$

Differentiate both members of this equation, then

$$\cos\theta = 1 - \frac{\theta^2}{2\,!} + \frac{\theta^4}{4\,!} - \frac{\theta^6}{6\,!} + \cdots\cdots. \qquad \text{Q. E. D.}$$

NOTE. That $\sin\theta = \mathrm{A} + \mathrm{B}\theta + \mathrm{C}\theta^2 + \mathrm{D}\theta^3 + \cdots\cdots$, as assumed in the above demonstration, is not self-evident, though true. Most functions can be developed in this form, but some cannot.

Cor. *If θ be any angle, then:*

96] $$\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{3.5} + \frac{17\theta^7}{3^2.5.7} + \frac{62\theta^9}{3^4.5.7} + \cdots\cdots;$$

97] $$\cot\theta = \frac{1}{\theta} - \frac{\theta}{3} - \frac{\theta^3}{3^2.5} - \frac{2\theta^5}{3^3.5.7} - \frac{\theta^7}{3^3.5^2.7} - \cdots\cdots;$$

98] $$\sec\theta = 1 + \frac{\theta^2}{2} + \frac{5\theta^4}{2^3.3} + \frac{61\theta^6}{2^4.3^2.5} + \frac{277\theta^8}{2^7.3^2.7} + \cdots\cdots;$$

99] $$\csc\theta = \frac{1}{\theta} + \frac{\theta}{2.3} + \frac{7\theta^3}{2^3.3^2.5} + \frac{31\theta^5}{2^4.3^3.5.7} + \frac{127\theta^7}{2^7.3^3.5^2.7} + \cdots\cdots$$

The reader may prove, as exercises, by either of two methods:

(1) *By Division:*

For the tangent, divide the development of the sine by that of the cosine;

for the cotangent, divide the development of the cosine by that of the sine;

for the secant, divide unity by the development of the cosine;

for the cosecant, divide unity by the development of the sine.

The work is facilitated by using detached coefficients.

(2) *By the method of unknown coefficients:*

Assume $\tan\theta, = \dfrac{\theta - \dfrac{\theta^3}{3!} + \dfrac{\theta^5}{5!} - \dfrac{\theta^7}{7!} + \cdots\cdots}{1 - \dfrac{\theta^2}{2!} + \dfrac{\theta^4}{4!} - \dfrac{\theta^6}{6!} + \cdots\cdots}, = \theta + \mathrm{A}\theta^3 + \mathrm{B}\theta^5 + \mathrm{C}\theta^7 + \cdots\cdots,$

and thence find A, B, C, ...

So, for cotangent, secant, and cosecant.

Again, $\because \mathrm{D}_\theta \log\sin\theta = \dfrac{\cos\theta}{\sin\theta} = \cot\theta = \dfrac{1}{\theta} - \dfrac{\theta}{3} - \dfrac{\theta^3}{3^2.5} - \cdots\cdots,$ [97

100] $$\therefore \log\sin\theta = \log\theta - \frac{\theta^2}{2.3} - \frac{\theta^4}{2^2.3^2.5} - \frac{\theta^6}{3^4.5.7} - \cdots\cdots,$$

i.e., the expression whose D_θ is the above series for $\cot\theta$.

So, $\because \mathrm{D}_\theta \log\cos\theta = -\dfrac{\sin\theta}{\cos\theta} = -\tan\theta = -\left(\theta + \dfrac{\theta^3}{3} + \cdots\cdots\right),$ [96

101] $$\therefore \log\cos\theta = -\left(\frac{\theta^2}{2} + \frac{\theta^4}{2^2.3} + \frac{\theta^6}{3^2.5} + \frac{17\theta^8}{2^3.3^2.5.7} + \cdots\cdots\right).$$

§ 7. THE TRIGONOMETRIC CANON.

The *Trigonometric Canon* is a set of tables which give the sine, cosine, tangent, cotangent, secant and cosecant for every angle, from 0 to a right angle, taken at regular intervals of say $10''$, or $1'$, or $10'$, ..., as may be chosen for the particular tables. Such tables contain either the functions themselves or their logarithms. The first are called the *natural* functions, and the others the *logarithmic* functions.

PROBLEM 1. TO CONSTRUCT A TABLE OF NATURAL SINES AND COSINES, TO MINUTES OF ANGLE.

FIRST METHOD: *Assume sin* $1'$ *as differing insensibly from arc* $1'$, *i.e.*, *that* $\sin 1' = .000\,290\,8882$,
and hence, *that* $\cos 1', = \sqrt{1-\sin^2 1'}, = .999\,999\,9577$; *then:*

(1) *For angles* 0°–30°, *apply formulae:*

$$\sin(\theta+\theta') = 2\sin\theta\cos\theta' - \sin(\theta-\theta'), \qquad [51$$

$$\cos(\theta+\theta') = 2\cos\theta\cos\theta' - \cos(\theta-\theta'); \qquad [53$$

make $\theta = 1', 2', 3', \ldots\ldots$ *successively, and* $\theta' = 1'$ *constantly*, thus·

$$\begin{aligned}\sin 2' &= 2\sin 1'\cos 1' - \sin 0'\\ &= 2\times .000\,290\,8882\times .999\,999\,9577 - 0\\ &= .000\,581\,7764\times(1-.000\,000\,0423)\\ &= .000\,581\,7764;\end{aligned}$$

$$\begin{aligned}\sin 3' &= 2\sin 2'\cos 1' - \sin 1'\\ &= 2\times .000\,581\,7764\times(1-.000\,000\,0423)\\ &\quad -.000\,290\,8882\\ &= .000\,872\,6646.\end{aligned}$$

So,

$$\begin{aligned}\cos 2' &= 2\cos 1'\cos 1' - \cos 0'\\ &= 2\times .999\,999\,9577\times(1-.000\,000\,0423) - 1\\ &= .999\,999\,8308;\end{aligned}$$

$$\begin{aligned}\cos 3' &= 2\cos 2'\cos 1' - \cos 1'\\ &= 2\times .999\,999\,8308\times(1-.000\,000\,0423)\\ &\quad -.999\,999\,9577\\ &= .999\,999\,6193.\end{aligned}$$

(2) *For angles* 30°–45°, *substitute* 30° *for* θ, *and* $1'$, $2'$, $3'$, ….. *successively for* θ' *in formulae:*

$$\begin{aligned}\sin(\theta+\theta') &= 2\sin\theta\cos\theta' - \sin(\theta-\theta') &&\text{[51}\\ &= \cos\theta' - \sin(\theta-\theta')\,; &&[\sin 30^\circ = \tfrac{1}{2}\\ \cos(\theta+\theta') &= \cos(\theta-\theta') - 2\sin\theta\sin\theta' &&\text{[54}\\ &= \cos(\theta-\theta') - \sin\theta'.\end{aligned}$$

$$\begin{aligned}\text{Thus: } \sin 30^\circ\,1' &= \cos 1' - \sin 29^\circ\,59'\\ &= .999\,999\cdots - .499\,75 = .500\,25\,;\\ \sin 30^\circ\,2' &= \cos 2' - \sin 29^\circ\,58'\\ &= .999\,999\cdots - .499\,50 = .500\,50.\\ \text{So, } \cos 30^\circ\,1' &= \cos 29^\circ\,59' - \sin 1'\\ &= .866\,17 - .000\,29 = .865\,88\,;\\ \cos 30^\circ\,2' &= \cos 29^\circ\,58' - \sin 2' = .865\,73.\end{aligned}$$

(3) *For angles* 45°–90°, *apply formulae:*

$$\begin{aligned}\sin(45^\circ+\theta') &= \cos(45^\circ-\theta')\,; &&\text{[I. Thm. 5}\\ \cos(45^\circ+\theta') &= \sin(45^\circ-\theta').\end{aligned}$$

$$\begin{aligned}\text{Thus: } \sin 45^\circ\,1' &= \cos 44^\circ\,59' = .707\,31\,;\\ \sin 45^\circ\,2' &= \cos 44^\circ\,58' = .707\,52.\\ \text{So, } \cos 45^\circ\,1' &= \sin 44^\circ\,59' = .706\,90\,;\\ \cos 45^\circ\,2' &= \sin 44^\circ\,58' = .706\,70.\end{aligned}$$

*SECOND METHOD: *Substitute the value of* θ *for* $0'$, $1'$, $2'$, ….. *in* [94, 95]. Thus:

$$\because\quad 1' = \frac{\pi}{180\times 60} = \frac{3.141\,592\,653\,589\,793}{10\,800}$$

$$= .000\,290\,888\,208\,666\,;$$

$$\therefore\ \sin 1' = .000\,290\,888\,208\,666 - \frac{.000\,290\,888\,208\,666^3}{3\,!} + \cdots\cdots$$

$$= .000\,290\,888\,2046\,;$$

$$\cos 1' = 1 - \frac{.000\,290\,888\,208\,666^2}{2\,!} + \cdots\cdots$$

$$= .999\,999\,9577\,;$$

$$\begin{aligned}\sin 2' &= 2 \times .000\,290\,888\,208\,666 \\ &\quad - 2^3 \times \frac{.000\,290\,888\,208\,666^3}{3\,!} + \cdots\cdots \\ &= .000\,581\,7764\,; \\ \cos 2' &= 1 - 2^2 \times \frac{.000\,290\,888\,208\,666^2}{2\,!} + \cdots\cdots \\ &= .999\,999\,8308.\end{aligned}$$

NOTE 1. The process is evidently very tedious; but the reader will notice, first, that it would be much shorter if four or five decimal places only were sought in the functions; and, second, that once having raised the fraction to the required powers, thereafter he has only to take simple multiples of them. At first he need use but two terms of the series; but later, when θ is larger, and the series therefore converges less rapidly, more terms must be used; thus:

$$\because, \text{ for } 30°, \ \theta = \frac{\pi}{6} = .52360 \text{ nearly};$$

$$\begin{aligned}\therefore \ \sin 30° &= .52360 - \frac{.5236^3}{6} + \frac{.5236^5}{120} - \frac{.5236^7}{5040} + \cdots\cdots \\ &= .52360 - .02392 + .00033 - .00000 + \cdots\cdots \\ &= .5 \text{ within less than } .00001\,;\end{aligned}$$

i.e., by the use of three terms of the series, the sine is found correct to four decimal places, the same degree of accuracy as that assumed for the value of π.

NOTE 2. The results may be verified by using both methods of computation; and for certain angles there are other and independent methods:

(1) $\because \ \sin \frac{1}{2}\theta = \sqrt{\dfrac{1 - \cos\theta}{2}}$, and $\cos \frac{1}{2}\theta = \sqrt{\dfrac{1 + \cos\theta}{2}}$; [63, 64

$\therefore$ from $\cos 45°, = \sqrt{\frac{1}{2}}$, [I. Thm. 13

are found in succession the sines and cosines of 22° 30′, 11° 15′, 5° 37′ 30″, …….

So, from $\cos 30°, = \frac{1}{2}\sqrt{3}$, [I. Thm. 12

are found in succession the sines and cosines of 15°, 7° 30′, 3° 45′, …….

(2) $\because\ \sin 2\theta = 2\sin\theta\cos\theta,$ [60

and $\cos 3\theta = 4\cos^3\theta - 3\cos\theta,$ [84

and $\because\ \sin 36° = \cos 54°,$ [I. Thm. 5

$\therefore\ 2\sin 18°\cos 18° = 4\cos^3 18° - 3\cos 18°\,;$

$\therefore\ \sin 18° = \frac{1}{4}(\sqrt{5} - 1), \quad \text{and } \cos 18° = \frac{1}{4}\sqrt{(10 + 2\sqrt{5})}\,;$

whence are found in succession the sines and cosines of 9°, 4° 30′, 2° 15′, ...

(3) From $\cos 36° = \cos^2 18° - \sin^2 18° = \frac{1}{4}(\sqrt{5} + 1),$ [61, above

and $\sin 36° = \sqrt{(1 - \cos^2 36°)} = \frac{1}{4}\sqrt{(10 - 2\sqrt{5})},$ [36, above

are found the sine and cosine of (36° − 30°), *i.e.*, of 6°, and thence in succession the sine and cosine of 3°, 1° 30′, 45′, ...

(4) From $\sin(36° + \theta') - \sin(36° - \theta'), = 2\cos 36°\sin\theta'$ [52

$= \frac{1}{2}(\sqrt{5} + 1)\sin\theta',$ [above

subtract $\sin(72° + \theta') - (\sin 72° - \theta'), = 2\cos 72°\sin\theta'$ [52

$= \frac{1}{2}(\sqrt{5} - 1)\sin\theta'.$ [above

then, 102]

$$\sin(36° + \theta') - \sin(36° - \theta') = \sin(72° + \theta') - \sin(72° - \theta') + \sin\theta',$$

which is Euler's formula of verification, and serves to test the sines of all angles from 0° to 90°, if to θ' be given the different values from 0° to 18°.

PROB. 2. TO COMPUTE TANGENTS, COTANGENTS, SECANTS AND COSECANTS.

FIRST METHOD. *For the tangents, divide the sines of the angles, in order, by the cosines, each by each; for the cotangents, divide the cosines by the sines; for the secants, divide unity by the cosines; for the cosecants, divide unity by the sines.*

*SECOND METHOD. *Substitute the values of* θ *for* 1′, 2′, 3′, ... *in formulae* [96–99].

NOTE. These series converge less rapidly than those for sine and cosine; but for small angles they may be used.

Prob. 3. To compute tables of logarithmic functions:

First Method. *From a table of logarithms of numbers take out the logarithms of the natural sines and cosines. For the tangents, subtract the logarithmic cosines from the logarithmic sines; for the cotangents, subtract the logarithmic sines from the logarithmic cosines; for the secants and cosecants, subtract the logarithmic cosines and sines from* 0, *respectively.*

*Second Method. *For the sines and cosines, substitute the values of θ for* 1′, 2′, 3′, ... *in formulae* [100, 101]; *for the tangents, cotangents, secants and cosecants, follow the first method.*

Note. A more rapid method, applicable also to making tables of natural functions, and many others, is this:

Compute the logarithms of three, four, or more angles at regular intervals, and find their several "orders of differences"; then, by the algebraic "method of differences," find the successive terms of the series of logarithms,
and interpolate for other angles lying between those of the series. Repeat this process for different parts of the table, and verify by direct computation. For safety, four-place tables must be computed to six places; five-place tables to seven places, and so on.

A useful modification of the above rule is this:

Add the last difference of the highest order to the last difference of the next lower order, and that sum to the last difference of the next lower order, and so on till a term of the series is reached. Thus, in the example which follows, the numbers below the heavy rules are got by successive addition:

Angle	log sine	1st difs.	2d difs.	3d difs.
18°	9.489 982 4	3 868 9		
18° 10′	9.493 851 3	3 831 1	−378	7
18° 20′	9.497 682 4	3 794 0	−371	7
18° 30′	9.501 476 4	3 757 6	−364	7
18° 40′	9.505 234 0	3 721 9	−357	7
18° 50′	9.508 955 9	3 686 9	−350	
19°	9.512 642 8			

*§ 8. EXERCISES.

If A, B, C be any three plane angles whose sum is 180° (the three angles of a triangle), prove that:

1. $\tan A + \tan B + \tan C = \tan A \tan B \tan C$. [80

2. $\tan\frac{1}{2}A \tan\frac{1}{2}B + \tan\frac{1}{2}B \tan\frac{1}{2}C + \tan\frac{1}{2}C \tan\frac{1}{2}A = 1$. [80

3. $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$. [68, 10, 71, 4

4. $\sin A + \sin B + \sin C = 4 \cos\frac{1}{2}A \cos\frac{1}{2}B \cos\frac{1}{2}C$. [68, 4, 70

Prove that:

5. $\sin(2\sin^{-1}x) = 2x\sqrt{(1-x^2)}$. [60

6. $\sin(3\sin^{-1}x) = 3x - 4x^3$. [78

7. $\tan(2\tan^{-1}x) = 2x : (1-x^2)$. [62

8. $\tan(3\tan^{-1}x) = (3x - x^3) : (1 - 3x^2)$. [80

9. $\sin^{-1}x \pm \sin^{-1}y = \sin^{-1}[x\sqrt{(1-y^2)} \pm y\sqrt{(1-x^2)}]$. [39

10. $\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}[xy \mp \sqrt{(1-x^2-y^2+x^2y^2)}]$. [41

11. $\tan^{-1}x \pm \tan^{-1}y = \tan^{-1}(x \pm y) : (xy \mp 1)$. [43, 44

12. $\frac{1}{2}\pi = \sin^{-1}\frac{3}{5} + \sin^{-1}\frac{4}{5}$. [39

13. $\frac{1}{4}\pi = \tan^{-1}1 = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$ [43

$= \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8} = 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7}$. [80

14. $\tan 3\theta = \dfrac{\sin\theta + \sin 3\theta + \sin 5\theta}{\cos\theta + \cos 3\theta + \cos 5\theta}$. [39, 41

15. If $(1 + \epsilon\cos\theta)(1 - \epsilon\cos\phi) = 1 - \epsilon^2$

then also, $\tan\frac{1}{2}\theta : \tan\frac{1}{2}\phi = \sqrt{(1+\epsilon)} : \sqrt{(1-\epsilon)}$. [65

16. If θ, θ' and θ'' be any three angles, prove that

$$\left.\begin{array}{l} -\sin(\theta+\theta'+\theta'')+\sin(-\theta+\theta'+\theta'') \\ +\sin(\theta-\theta'+\theta'')+\sin(\theta+\theta'-\theta'') \end{array}\right\} = 4\sin\theta\sin\theta'\sin\theta'', \quad [78$$

$$\left.\begin{array}{l} \cos(\theta+\theta'+\theta'')+\cos(-\theta+\theta'+\theta'') \\ +\cos(\theta-\theta'+\theta'')+\cos(\theta+\theta'-\theta'') \end{array}\right\} = 4\cos\theta\cos\theta'\cos\theta''. \quad [79$$

17. If θ and θ' be any two angles, and n any integer, prove that

$$[\sin\theta + \sin(\theta+\theta') + \sin(\theta+2\theta') + \cdots + \sin(\theta+\overline{n-1}\,\theta')] \,.\, 2\sin\tfrac{1}{2}\theta'$$
$$= \cos(\theta - \tfrac{1}{2}\theta') - \cos(\theta + \overline{n-\tfrac{1}{2}}\,\theta'), \quad [54$$

$$[\cos\theta + \cos(\theta+\theta') + \cos(\theta+2\theta') + \cdots + \cos(\theta+\overline{n-1})\theta'] \,.\, 2\sin\tfrac{1}{2}\theta'$$
$$= \sin(\theta + \overline{n-\tfrac{1}{2}}\,\theta') - \sin(\theta - \tfrac{1}{2}\theta'). \quad [52$$

18. From the results of Ex. 17, prove that

$$\sin\theta + \sin\left(\theta + \frac{2\pi}{n}\right) + \sin\left(\theta + \frac{4\pi}{n}\right) + \cdots + \sin\left(\theta + \frac{2(n-1)\pi}{n}\right) = 0,$$

$$\cos\theta + \cos\left(\theta + \frac{2\pi}{n}\right) + \cos\left(\theta + \frac{4\pi}{n}\right) + \cdots\cdots + \cos\left(\theta + \frac{2(n-1)\pi}{n}\right) = 0,$$

wherein n is any positive integer.

19. In the results of Ex. 18, make $n = 3$, and prove that

$$\sin\theta + \sin(60^\circ - \theta) - \sin(60^\circ + \theta) = 0\,;$$
$$\cos\theta - \cos(60^\circ - \theta) - \cos(60^\circ + \theta) = 0.$$

20. In the results of Ex. 18, make $n = 5$, and prove that

$$\sin\theta + \sin(72^\circ + \theta) + \sin(36^\circ - \theta) - \sin(36^\circ + \theta) - \sin(72^\circ - \theta) = 0,$$
$$\cos\theta + \cos(72^\circ + \theta) - \cos(36^\circ - \theta) - \cos(36^\circ + \theta) + \cos(72^\circ - \theta) = 0,$$

and from the first of these two, get Euler's formula. [102

21. In the results of Ex. 18, make $n = 9$ and 15, in succession, and thence find other formulae of verification.

Show that the formula found when $n = 9$ verifies the sines of all angles in the quadrant, if to θ be given values from 0° to 10°.

22. If θ be any angle and θ' be an infinitesimal angle, the increment of θ, prove that

$$\text{inc}^2 \sin\theta = -\left(2\sin\tfrac{1}{2}\theta'\right)^2 \sin(\theta + \theta'), \qquad [69$$
$$\text{inc}^2 \cos\theta = -\left(2\sin\tfrac{1}{2}\theta'\right)^2 \cos(\theta + \theta'), \qquad [71$$
$$\text{inc}^4 \sin\theta = \quad \left(2\sin\tfrac{1}{2}\theta'\right)^4 \sin(\theta + 2\theta'),$$
$$\text{inc}^4 \cos\theta = \quad \left(2\sin\tfrac{1}{2}\theta'\right)^4 \cos(\theta + 2\theta'),$$

wherein $\text{inc}^2 \sin\theta$ stands for the increment of the increment of $\sin\theta$,

i.e., for $[\sin(\theta + 2\theta') - \sin(\theta + \theta')] - [\sin(\theta + \theta') - \sin\theta]$,

or $\sin(\theta + 2\theta') - 2\sin(\theta + \theta') + \sin\theta$,

and $\text{inc}^4\theta$ stands for inc inc inc inc $\sin\theta$, *i.e.*, for

$\sin(\theta + 4\theta') - 4\sin(\theta + 3\theta') + 6\sin(\theta + 2\theta') - 4\sin(\theta + \theta') + \sin\theta$

and so on. [Alg., Meth. Dif.

23. Compute the sines and cosines

of 22° 30′ and 67° 30′, 11° 15′ and 78° 45′, ... ;
of 15° and 75°, 7° 30′ and 82° 30′, ... ;
of 6° and 84°, 3° and 87°, ... ;
of 9° and 81°, 4° 30′ and 85° 30′, ...

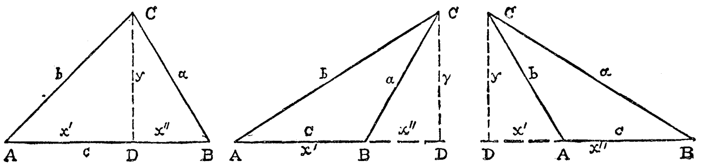
[Prob. 1, note 2

24. From the logarithmic sines of 18°, 18° 10′, 18° 20′, ... find the several orders of differences, and thence, by interpolation, find the logarithmic sines of 18° 1′, 18° 2′, 18° 3′, 18° 4′, ...

III. SOLUTION OF PLANE TRIANGLES.

§ 1. GENERAL PROPERTIES OF PLANE TRIANGLES.

THEOREM 1. *In any plane triangle the sides are proportional to the sines of the opposite angles.*



Let ABC be any triangle; a, b, c the sides opposite the angles A, B, C respectively; then will:

103] $a : b = \sin A : \sin B$;

$b : c = \sin B : \sin C$;

$c : a = \sin C : \sin A$.

For, draw $DC \perp AB$; then

$\because$ $DC = AC \sin A = b \sin A$, [I. § 7, note

and $DC = BC \sin B = a \sin B$,

$\therefore$ $b \sin A = a \sin B$,

$\therefore$ $a : b = \sin A : \sin B$. [Thm. prop'n

So, $b : c = \sin B : \sin C$,

and $c : a = \sin C : \sin A$. Q. E. D.

NOTE 1. This theorem may be stated more symmetrically thus:

$$a : b : c = \sin A : \sin B : \sin C;$$

or thus:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Cor. *In any plane triangle* ABC :

104]
$$a = \frac{b \sin \mathrm{A}}{\sin \mathrm{B}} = \frac{c \sin \mathrm{A}}{\sin \mathrm{C}};$$
$$b = \frac{c \sin \mathrm{B}}{\sin \mathrm{C}} = \frac{a \sin \mathrm{B}}{\sin \mathrm{A}};$$
$$c = \frac{a \sin \mathrm{C}}{\sin \mathrm{A}} = \frac{b \sin \mathrm{C}}{\sin \mathrm{B}};$$

105]
$$\sin \mathrm{A} = \frac{a \sin \mathrm{B}}{b} = \frac{a \sin \mathrm{C}}{c};$$
$$\sin \mathrm{B} = \frac{b \sin \mathrm{C}}{c} = \frac{b \sin \mathrm{A}}{a};$$
$$\sin \mathrm{C} = \frac{c \sin \mathrm{A}}{a} = \frac{c \sin \mathrm{B}}{b}.$$

Note 2. The expressions $\frac{a}{\sin \mathrm{A}}$, ... are equal to the diameter of the circumscribed circle, as appears later, Prob. 4.

Note 3. All the angles of a triangle are positive angles, and the sides of the triangle are positive lines when taken with reference to the angles included by them.
The ordinate of the vertex, with reference to the base, is therefore always positive ;
but the abscissa may be either positive or negative, according as it is measured in a positive or negative direction from the origin used.

Thm. 2. *In any plane triangle, the sum of any two sides is to their difference as the tangent of half the sum of the two opposite angles is to the tangent of half their difference :*

106] *i.e.*, $(a + b) : (a \sim b) = \tan \frac{1}{2}(\mathrm{A} + \mathrm{B}) : \tan \frac{1}{2}(\mathrm{A} \sim \mathrm{B})$;
$(b + c) : (b \sim c) = \tan \frac{1}{2}(\mathrm{B} + \mathrm{C}) : \tan \frac{1}{2}(\mathrm{B} \sim \mathrm{C})$;
$(c + a) : (c \sim a) = \tan \frac{1}{2}(\mathrm{C} + \mathrm{A}) : \tan \frac{1}{2}(\mathrm{C} \sim \mathrm{A})$.

For, $\because a : b = \sin \mathrm{A} : \sin \mathrm{B}$, [Thm. 1
$\therefore (a + b) : (a - b) = (\sin \mathrm{A} + \sin \mathrm{B}) : (\sin \mathrm{A} - \sin \mathrm{B})$; [Thm. prop'n
and $\because (\sin \mathrm{A} + \sin \mathrm{B}) : (\sin \mathrm{A} - \sin \mathrm{B}) = \tan \frac{1}{2}(\mathrm{A} + \mathrm{B}) : \tan \frac{1}{2}(\mathrm{A} - \mathrm{B})$; [72
$\therefore (a + b) : (a - b) = \tan \frac{1}{2}(\mathrm{A} + \mathrm{B}) : \tan \frac{1}{2}(\mathrm{A} - \mathrm{B})$.

But $\because$ the greater side of a triangle lies opposite the greater angle, [*geom.*

$\therefore$ when $a > b$, then also $\mathrm{A} > \mathrm{B}$,

and $(a+b) : (a-b) = \tan\frac{1}{2}(\mathrm{A}+\mathrm{B}) : \tan\frac{1}{2}(\mathrm{A}-\mathrm{B})$.

So, when $b > a$, then also $\mathrm{B} > \mathrm{A}$,

and $(a+b) : (b-a) = \tan\frac{1}{2}(\mathrm{A}+\mathrm{B}) : \tan\frac{1}{2}(\mathrm{B}-\mathrm{A})$;

$\therefore (a+b) : (a \sim b) = \tan\frac{1}{2}(\mathrm{A}+\mathrm{B}) : \tan\frac{1}{2}(\mathrm{A} \sim \mathrm{B})$.

So, $(b+c) : (b \sim c) = \tan\frac{1}{2}(\mathrm{B}+\mathrm{C}) : \tan\frac{1}{2}(\mathrm{B} \sim \mathrm{C})$,

and $(c+a) : (c \sim a) = \tan\frac{1}{2}(\mathrm{C}+\mathrm{A}) : \tan\frac{1}{2}(\mathrm{C} \sim \mathrm{A})$. Q. E. D.

COR. *In any plane triangle* ABC :

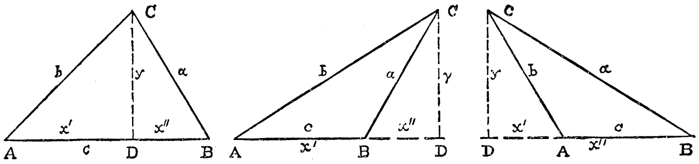
107] $$\tan\tfrac{1}{2}(\mathrm{A} \sim \mathrm{B}) = \frac{a \sim b}{a+b} \cdot \tan\tfrac{1}{2}(\mathrm{A}+\mathrm{B}) = \frac{a \sim b}{a+b} \cot\tfrac{1}{2}\mathrm{C};$$

$$\tan\tfrac{1}{2}(\mathrm{B} \sim \mathrm{C}) = \frac{b \sim c}{b+c} \cdot \tan\tfrac{1}{2}(\mathrm{B}+\mathrm{C}) = \frac{b \sim c}{b+c} \cot\tfrac{1}{2}\mathrm{A};$$

$$\tan\tfrac{1}{2}(\mathrm{C} \sim \mathrm{A}) = \frac{c \sim a}{c+a} \cdot \tan\tfrac{1}{2}(\mathrm{C}+\mathrm{A}) = \frac{c \sim a}{c+a} \cot\tfrac{1}{2}\mathrm{B}.$$

THM. 3. *In any plane triangle* ABC :

108] $$\cos\mathrm{A} = \frac{b^2+c^2-a^2}{2bc}; \quad \cos\mathrm{B} = \frac{c^2+a^2-b^2}{2ca}; \quad \cos\mathrm{C} = \frac{a^2+b^2-c^2}{2ab}.$$



For, $\because$ $a^2 = b^2 + c^2 - 2c \cdot \mathrm{AD}$, [*geom.*

and $\mathrm{AD} = b\cos\mathrm{A}$; [I. § 7, note

$$\therefore \quad a^2 = b^2 + c^2 - 2bc\cos\mathrm{A},$$

and $$\cos\mathrm{A} = \frac{b^2+c^2-a^2}{2bc}.$$

So, $$\cos\mathrm{B} = \frac{c^2+a^2-b^2}{2ca},$$

and $$\cos\mathrm{C} = \frac{a^2+b^2-c^2}{2ab}.$$ Q. E. D.

THM. 4. *In any plane triangle* ABC:

109] $$\sin\tfrac{1}{2}\text{A} = \sqrt{\frac{(s-b)(s-c)}{bc}},$$

$$\sin\tfrac{1}{2}\text{B} = \sqrt{\frac{(s-c)(s-a)}{ca}},$$

$$\sin\tfrac{1}{2}\text{C} = \sqrt{\frac{(s-a)(s-b)}{ab}},$$

wherein s stands for $\frac{1}{2}(a+b+c)$, or half the perimeter.

For, $\because$ $$2\sin^2\tfrac{1}{2}\text{A} = 1 - \cos\text{A} \qquad [63$$

$$= 1 - \frac{b^2+c^2-a^2}{2bc} \qquad [\text{Thm. } 3$$

$$= \frac{a^2-(b-c)^2}{2bc}$$

$$= \frac{(a-b+c)(a+b-c)}{2bc}$$

$$= \frac{4(s-b)(s-c)}{2bc};$$

$$\therefore \quad \sin\tfrac{1}{2}\text{A} = \sqrt{\frac{(s-b)(s-c)}{bc}}.$$

So, $$\sin\tfrac{1}{2}\text{B} = \sqrt{\frac{(s-c)(s-a)}{ca}};$$

and $$\sin\tfrac{1}{2}\text{C} = \sqrt{\frac{(s-a)(s-b)}{ab}}.$$ Q. E. D.

THM. 5. *In any plane triangle* ABC:

110] $$\cos\tfrac{1}{2}\text{A} = \sqrt{\frac{s(s-a)}{bc}};$$

$$\cos\tfrac{1}{2}\text{B} = \sqrt{\frac{s(s-b)}{ca}};$$

$$\cos\tfrac{1}{2}\text{C} = \sqrt{\frac{s(s-c)}{ab}}.$$

For, $\because$ $2\cos^2\frac{1}{2}A = 1 + \cos A$ [64

$$= 1 + \frac{b^2 + c^2 - a^2}{2bc} \quad \text{[Thm. 3}$$

$$= \frac{(b+c)^2 - a^2}{2bc}$$

$$= \frac{(b+c+a)(b+c-a)}{2bc}$$

$$= \frac{4s(s-a)}{2bc};$$

$$\therefore \quad \cos\tfrac{1}{2}A = \sqrt{\frac{s(s-a)}{bc}}.$$

So, $\cos\frac{1}{2}B = \sqrt{\frac{s(s-b)}{ca}}$;

and $\cos\frac{1}{2}C = \sqrt{\frac{s(s-c)}{ab}}$. Q. E. D.

THM. 6. *In any plane triangle* ABC:

111] $$\tan\tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}};$$

$$\tan\tfrac{1}{2}B = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}};$$

$$\tan\tfrac{1}{2}C = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}.$$

For, $\tan\frac{1}{2}A = \sin\frac{1}{2}A : \cos\frac{1}{2}A$ [35

$$= \sqrt{\frac{(s-b)(s-c)}{bc}} : \sqrt{\frac{s(s-a)}{bc}} \quad \text{[Thm. 4, 5}$$

$$= \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

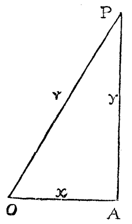
So, $\tan\frac{1}{2}B = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$;

and $\tan\frac{1}{2}C = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$. Q. E. D.

NOTE. Since no angle is greater than 180°, therefore no half-angle is greater than 90°, and the radicals of Thms. 4, 5, and 6 are all positive. [I. Thm. 3

§ 2. SOLUTION OF RIGHT TRIANGLES.

PROBLEM 1. TO SOLVE A RIGHT TRIANGLE:



Let OAP be any right triangle; O, the base-angle; A, the right angle; x, y, and r, the base, perpendicular and hypothenuse.

CASE 1. *Given the hypothenuse and an acute angle*, for example r and O; then:

$\mathrm{P} = 90^\circ - \mathrm{O}$;
$x = r \cos \mathrm{O}$, or $= r \sin \mathrm{P}$;
$y = r \sin \mathrm{O}$, or $= r \cos \mathrm{P}$, or $= x \tan \mathrm{O}$. [I. § 7

CASE 2. *Given a side and an acute angle*, for example x and O; then:

$\mathrm{P} = 90^\circ - \mathrm{O}$;
$r = x : \cos \mathrm{O}$, or $= x : \sin \mathrm{P}$;
$y = x \tan \mathrm{O}$, or $= r \sin \mathrm{O}$, or $= r \cos \mathrm{P}$. [I. § 7

CASE 3. *Given the hypothenuse and a side*, for example r and x; then:

$\cos \mathrm{O} = x : r$, whence O is found;
$\mathrm{P} = 90^\circ - \mathrm{O}$;
$y = r \sin \mathrm{O}$, or $= r \cos \mathrm{P}$, or $= x \tan \mathrm{O}$, [I. § 7
or $= \sqrt{r^2 - x^2} = \sqrt{(r + x)(r - x)}$. [*geom.*

CASE 4. *Given the two sides about the right angle*, then:

$\tan \mathrm{O} = y : x$, whence O is found;
$\mathrm{P} = 90^\circ - \mathrm{O}$;
$r = x : \cos \mathrm{O}$, or $= x : \sin \mathrm{P}$, or $= y : \sin \mathrm{O}$,
or $= y : \cos \mathrm{P}$, [I. § 7
or $= \sqrt{x^2 + y^2}$. [*geom.*

NOTE 1. To test the correctness of the work, various checks may be applied:

(1) Compute the part by some other process.

(2) Substitute the three computed parts in some one of the six equations which express the definitions of the trigonometric functions [I. § 7]; if this equation is satisfied the parts are right.

functions [I. § 7], or of the six equations which result therefrom [I. § 7, note]; if this equation is satisfied the parts are right.

(3) Take the three given parts and one computed part,
or two given parts and two computed parts,
or one given part and the three computed parts,
and substitute them in some one of the formulae of Thms. 1–6; if this formula is satisfied, the work is correct, for the part or parts tested.

*NOTE 2. When the solution involves angles near to 0°, 90° or 180°, care must be used in the selection of formulae; for, of angles near 90°, those which differ very considerably have nearly the same sine, and cannot therefore be determined with precision from the table of sines, thus:
the sines of all angles from 89° 50′ to 90°, inclusive, differ from 1 by less than .000005, and the sines of all angles from 89° 49′ to 89° 42′, inclusive, differ from .99999 by less than .000005;
but, of tangents, that of 89° is 57.2900; that of 89° 20′ is 85.9398; that of 89° 40′ is 171.885; that of 90° is infinity;
whereby it appears that the tangents of angles near 90° not only increase very fast, but that they also increase faster and faster.

So, of angles near 0° and 180°, the natural cosines change very slowly, and the natural cotangents very fast.

Of angles near 0° and 180°, the logarithmic sines, tangents and cotangents change very fast, and at rapidly varying rates, and the logarithmic cosines very slowly.

So, of angles near 90°, the logarithmic cosines, cotangents and tangents change very fast, and at rapidly varying rates, and the logarithmic sines very slowly.

To avoid these angles, use the following formulae:

(1) If x be very small compared with r, and therefore o be nearly 90°, use:

112] $$\sin \tfrac{1}{2}\mathrm{o} = \sqrt{\frac{r - x}{2r}};$$

113] $$\cos \tfrac{1}{2}\mathrm{o} = \sqrt{\frac{r + x}{2r}};$$

114] $\tan\frac{1}{2}\mathrm{O} = \sqrt{\dfrac{r-x}{r+x}} = \dfrac{y}{r+x} = \dfrac{r-x}{y}.$

For, ∵ $\cos\mathrm{O} = \dfrac{x}{r},$ [I. § 7

∴ $2\sin^2\frac{1}{2}\mathrm{O}, \;= 1 - \cos\mathrm{O}, \;= 1 - \dfrac{x}{r} \;= \dfrac{r-x}{r},$ [63

and $2\cos^2\frac{1}{2}\mathrm{O}, \;= 1 + \cos\mathrm{O}, \;= 1 + \dfrac{x}{r} \;= \dfrac{r+x}{r},$ [64

∴ etc.   Q. E. D.

(2) If x be nearly equal to r, and therefore O be nearly 0, compute $y, = \sqrt{(r+x)(r-x)}$, and use:

115] $\sin\frac{1}{2}\mathrm{P} = \sqrt{\dfrac{r-y}{2r}};$

116] $\cos\frac{1}{2}\mathrm{P} = \sqrt{\dfrac{r+y}{2r}};$

117] $\tan\frac{1}{2}\mathrm{P} = \sqrt{\dfrac{r-y}{r+y}} = \dfrac{x}{r+y} = \dfrac{r-y}{x}.$

(3) If x be very small as to y, and therefore O be nearly 90°, compute $r, = \sqrt{x^2+y^2}$, then use either of the formulae [112–114]; if y be very small as to x, use [115–117].

*NOTE 3. If O be very small, then special tables may be used which give the angle in seconds, the logarithmic sine and tangent, and the logarithms of the ratios of the sine and tangent to the number of seconds in the angle. These ratios are nearly constant, and their differences are very nearly proportional to the differences of the corresponding angles. The formulae are:

118] $\mathrm{O} \text{ in seconds} \,.\, \dfrac{\sin\mathrm{O}}{\mathrm{O}\text{ in seconds}} = \dfrac{y}{r};$

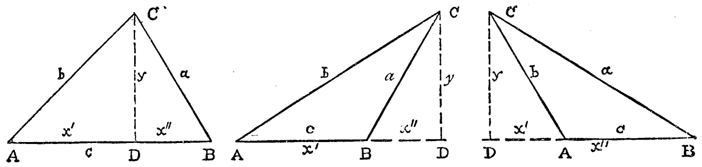
119]   and $\mathrm{O} \text{ in seconds} \,.\, \dfrac{\tan\mathrm{O}}{\mathrm{O}\text{ in seconds}} = \dfrac{y}{x}.$

So, if P be very small.

These special tables give the functions and angles with more accuracy than the ordinary tables give functions and angles in any part of the quadrant.

§ 3. SOLUTION OF OBLIQUE TRIANGLES.

PROB. 2. TO SOLVE AN OBLIQUE TRIANGLE.



FIRST METHOD. *By means of right triangles.*

Let ABC be any oblique triangle; and a, b, c the sides opposite the angles A, B, C, respectively.

From C draw DC $\perp$ AB, and let y stand for DC, x' for AD, x'' for DB, C$'$ for $\angle$ ACD, and C$''$ for $\angle$ DCB.

CASE 1. *Given two angles and a side;* for example, A, C and b; then:

$\text{B} = 180° - (\text{A} + \text{C})$; [*geom.*

In rt. $\triangle$ ACD, b and $\angle$ CAD are known, whence y and x' are found;

In rt. $\triangle$ BCD, y and $\angle$ CBD are known, whence a and x'' are found;

$c = x' + x''$.

NOTE 1. The reader must carefully distinguish between the signs of the sides and angles of a triangle, when taken with reference to the triangle itself, and when taken with reference only to some initial direction; thus:

The parts of the triangle BCD, when taken with reference to the triangle BCD, are all positive;

but, with reference to AB, as initial direction, the line DB is positive or negative according as it is measured to the right or to the left from D.

So, with reference to ordinary positive rotation, the angle DCB is positive or negative according as CB swings to the right or to the left from CD as the reader looks at the diagram.

NOTE 2. x' and C$'$ are positive or negative, with reference to AB and to ordinary positive rotation, according as A is acute or obtuse; and x'' and C$''$ are positive or negative, according as B is acute or obtuse.
If $A + C =$ or $> 180°$, there is no triangle;
if $A + C < 180$, there is always one triangle, and but one.
The parts are determined without ambiguity from the formulae.

CASE 2. *Given two sides and the included angle;* for example, b, c and A; then:

In rt. △ ACD, b and ∠ CAD are known, whence y, x' and C$'$ are found;

$x'' = c - x'$;

In rt. △ BCD, y and x'' are known, whence ∠ CBD and C$''$ are found;

$C = C' + C''$.

$B = \angle CBD$ or $= 180° - CBD$.

NOTE. x' and C$'$ are positive or negative, with reference to AB and to ordinary positive rotation, according as A is acute or obtuse,
and x'' and C$''$ are positive or negative, and B is acute or obtuse, according as $c >$ or $< x'$.
There is always one triangle and but one;
the parts are determined without ambiguity from the formulae.

CASE 3. *Given the three sides;* then:

$\because\ a^2 = y^2 + x''^2$ and $b^2 = y^2 + x'^2$, [*geom.*

$\therefore\ a^2 - b^2 = x''^2 - x'^2$.

$\therefore\ (a + b)(a - b) = (x'' + x'$, or $c)(x'' - x')$, whence $x'' - x'$ is found,

and $x' = \frac{1}{2}c - \frac{1}{2}(x'' - x')$, $x'' = \frac{1}{2}c + \frac{1}{2}(x'' - x')$.

In rt. △ ACD, b and x' are known, whence ∠ CAD is found;

In rt. △ BCD, a and x'' are known, whence ∠ CBD is found;

$C = 180° - (A + B)$.

NOTE. A and B are acute or obtuse, *i.e.*, equal or supplementary to CAD and CBD, according as x' and x'' are respectively positive or negative with reference to AB.

There is no triangle if either side equals or exceeds the sum of the other two; otherwise, there is one triangle, and but one. The parts are determined without ambiguity from the formulae.

CASE 4. *Given two sides and an angle opposite one of them;* for example, a, b and A.

In rt. $\triangle$ ACD, b and $\angle$ CAD are known, whence y, x' and C$'$ are found;

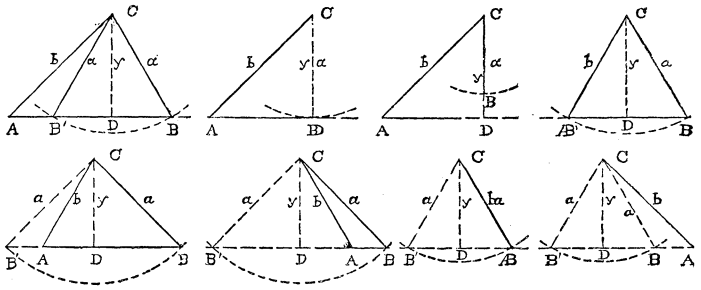
In rt. $\triangle$ BCD, a and y are known, whence x'', $\angle$ CBD and C$''$ are found;

$c = x' + x''$;

$\text{C} = \text{C}' + \text{C}''$;

$\text{B} = \angle \text{CBD}$ or $= 180° - \text{CBD}$, according as C$''$ is added to, or subtracted from, C$'$.

NOTE. There may be one triangle, two triangles, or none.



(1) If A is acute and $a < b$, but $a > y$, there are two triangles;
(2) if A is acute and $a < b$, but $a = y$, there is one triangle;
(3) if A is acute and $a < b$, but $a < y$, there is no triangle.

(4) If A is acute and $a = b$, there is one triangle;
(5) if A is acute and $a > b$, there is one triangle.

(6) If A is right or obtuse and $a > b$, there is one triangle;
(7) if A is right or obtuse and $a = b$, there is no triangle;
(8) if A is right or obtuse and $a < b$, there is no triangle.

For $\because$ C'' is found from its cosine, $= y : a$,

$\therefore$ when $a < b$, C'' and x'' may be either positive or negative; and $\text{B} = \text{CBD}$ or $= 180° - \text{CBD}$. [2

But, when $a =$ or $> b$, the negative values of x'' and C'' are inadmissible; for, then $c, = x' + {}^{-}x''$, and $\text{C}, = \text{C}' + {}^{-}\text{C}''$, are both 0 or negative, which is absurd.

SECOND METHOD. *By means of the general properties.*

CASE 1. *Given two angles and a side;* for example, A, B and c, then:

$$\text{C} = 180° - (\text{A} + \text{B}), \qquad [geom.$$

$$a = \frac{c}{\sin \text{C}} \sin \text{A}, \qquad b = \frac{c}{\sin \text{C}} \sin \text{B}. \qquad [104$$

Check: See formulae [120, 121].

NOTE. The formulae give one value, and but one, for each part.

CASE 2. *Given two sides and the included angle;* for example, a, b and C; then:

$$\tan \tfrac{1}{2}(\text{A} \sim \text{B}) = \frac{a \sim b}{a + b} \cot \tfrac{1}{2}\text{C}, \text{ whence } \tfrac{1}{2}(\text{A} \sim \text{B}) \text{ is found}, \qquad [107$$

$$\tfrac{1}{2}(\text{A} + \text{B}) + \tfrac{1}{2}(\text{A} \sim \text{B}) = \text{the greater of the two angles};$$

$$\tfrac{1}{2}(\text{A} + \text{B}) - \tfrac{1}{2}(\text{A} \sim \text{B}) = \text{the less of the two angles}.$$

$$c = \frac{a}{\sin \text{A}} \sin \text{C}. \qquad [104$$

Check: $b : \sin \text{B} = a : \sin \text{A}$.

NOTE. The formulae give one value, and but one, for each part.

CASE 3. *Given the three sides:* a, b, c.

Apply the formulae of Thms. 3–6.

Check: $\text{A} + \text{B} + \text{C} = 180°$.

NOTE 1. The formulae give one value, and but one, for each part.

NOTE 2. Of these formulae, those of Thm. 3 are only useful when the computation is by natural functions;
those of Thm. 4 use nine different logarithms,
those of Thm. 5 use ten different logarithms,
those of Thm. 6 use only seven different logarithms,
for the computation of all the angles. The formulae of Thm. 6

are therefore generally to be preferred; they may be put in the form:

$$\tan\tfrac{1}{2}\text{A} = \frac{1}{s-a}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}},$$

$$\tan\tfrac{1}{2}\text{B} = \frac{1}{s-b}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}},$$

$$\tan\tfrac{1}{2}\text{C} = \frac{1}{s-c}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}};$$

wherein the second factor of the right member is the same, and may be computed but once, for all. It will appear later, PROB. 4, that this factor is the radius of the inscribed circle.

CASE 4. *Given two sides and an angle opposite one of them;* for example, a, b and A; then:

$$\sin\text{B} = \frac{b}{a}\sin\text{A}, \text{ whence B is found.} \qquad [104$$

$$\text{C} = 180° - (\text{A} + \text{B}), \qquad [\textit{geom.}$$

$$c = \frac{a}{\sin\text{A}}\sin\text{C}. \qquad [104$$

Check: See formulae [120, 121].

NOTE 1. The formulae leave the parts in doubt;
for the same value of sin B belongs to two angles, which are supplements of each other;
so that, in general, B may be an acute or an obtuse angle;
whence two values each for C and c, and two triangles.
But this is limited by the conditions, that "the greater side of a triangle lies opposite the greater angle," and that "a triangle can have but one obtuse angle, and no side 0 or negative."
If, then, $a =$ or $> b$, B cannot be obtuse;
and if A is obtuse, B is acute.
Moreover, the shortest length possible for a is the perpendicular DC;
for $\because$ $b \sin\text{A} = a \sin\text{B} = \text{DC}$,
$\therefore$ if $a < \text{DC}$, then $\sin\text{B} > 1$, which is impossible.

NOTE 2. The same care must be taken in solving oblique triangles as in solving right triangles, to avoid the use of angles near 0°, 90° or 180°, unless the special table mentioned under right triangles is used.

The following formulae are useful; the reader may prove.

120] $$\frac{a+b}{c}=\frac{\cos\frac{1}{2}(\text{A}\sim\text{B})}{\sin\frac{1}{2}\text{C}}=\frac{\cos\frac{1}{2}(\text{A}\sim\text{B})}{\cos\frac{1}{2}(\text{A}+\text{B})};$$

121] $$\frac{a\sim b}{c}=\frac{\sin\frac{1}{2}(\text{A}\sim\text{B})}{\cos\frac{1}{2}\text{C}}=\frac{\sin\frac{1}{2}(\text{A}\sim\text{B})}{\sin\frac{1}{2}(\text{A}+\text{B})};$$

122] $$\frac{s}{c}=\frac{\cos\frac{1}{2}\text{A}\cos\frac{1}{2}\text{B}}{\sin\frac{1}{2}\text{C}};$$

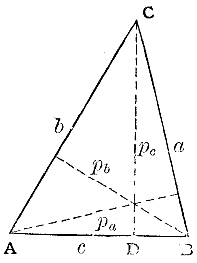
123] $$\frac{s-c}{c}=\frac{\sin\frac{1}{2}\text{A}\sin\frac{1}{2}\text{B}}{\sin\frac{1}{2}\text{C}};$$

124] $$\frac{s-b}{c}=\frac{\sin\frac{1}{2}\text{A}\cos\frac{1}{2}\text{B}}{\cos\frac{1}{2}\text{C}};$$

125] $$\frac{s-a}{c}=\frac{\cos\frac{1}{2}\text{A}\sin\frac{1}{2}\text{B}}{\cos\frac{1}{2}\text{C}}.$$

§ 4. THE AREA OF A TRIANGLE.

PROB. 3. TO FIND THE AREA OF A TRIANGLE, AND THE LENGTH OF THE PERPENDICULARS FROM THE VERTICES TO THE OPPOSITE SIDES:



Let ABC be any triangle, and let K stand for its area, and p_a, p_b, p_c, for the perpendiculars on a, b, c, respectively.

CASE 1. *Given two sides and the included angle;* for example, b, c, and A.

(1) *For the area, multiply half the product of any two sides by the sine of the included angle.*

(2) *For the perpendicular upon either given side, multiply the adjacent side by the sine of the included angle.*

For, draw DC ⊥ AB; then,

$\because$ $\text{K}=\frac{1}{2}\text{DC}\,.\,\text{AB}=\frac{1}{2}p_c c$; [*geom.*

and $\because$ $p_c=b\sin\text{A}$; [I. § 7, note

126] $\therefore$ $\text{K}=\frac{1}{2}bc\sin\text{A}$.

So, $\text{K}=\frac{1}{2}ca\sin\text{B}$;

and $\text{K}=\frac{1}{2}ab\sin\text{C}$.

CASE 2. *Given the angles and one side;* for example, c:

(1) *For the area, multiply half the square of any side by the sines of the adjacent angles, and divide the product by the sine of the opposite angle.*

(2) *For the perpendicular, multiply the side by the sines of the adjacent angles, and divide the product by the sine of the opposite angle.*

For, $\because\ K = \frac{1}{2} bc \sin A,$ [126

and $\because\ b = \dfrac{c \sin B}{\sin C},$ [104

127] $\therefore\ K = \dfrac{\frac{1}{2} c^2 \sin A \sin B}{\sin C}.$

So, $K = \dfrac{\frac{1}{2} a^2 \sin B \sin C}{\sin A},$ and $K = \dfrac{\frac{1}{2} b^2 \sin C \sin A}{\sin B}.$

128] $p_a = \dfrac{2K}{a} = \dfrac{a \sin B \sin C}{\sin A}.$

So, $p_b = \dfrac{b \sin C \sin A}{\sin B},$ and $p_c = \dfrac{c \sin A \sin B}{\sin C}.$

CASE 3. *Given the three sides, a, b, c:*

(1) *For the area, from half the sum of the sides subtract each side separately, multiply the continued product of these remainders by the half sum, and take the square root of the product.*

(2) *For the perpendicular, divide twice the root above found by the side on which the perpendicular falls.*

For, $\because\ K = \frac{1}{2} ab \sin C;$ [126

and $\because\ \sin C = 2 \sin \frac{1}{2} C \cos \frac{1}{2} C$ [60

$= 2\sqrt{\dfrac{(s-a)(s-b)}{ab}} \cdot \sqrt{\dfrac{s(s-c)}{ab}}$ [109, 110

$= \dfrac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)};$

129] $\therefore\ K = \sqrt{s(s-a)(s-b)(s-c)},$

130] $\therefore\ p_a = \dfrac{2K}{a} = \dfrac{2\sqrt{s(s-a)(s-b)(s-c)}}{a}.$

So, $p_b = \dfrac{2K}{b},$ and $p_c = \dfrac{2K}{c}.$

NOTE. The reader may prove the check-formula

131] $$\frac{1}{p_a}+\frac{1}{p_b}+\frac{1}{p_c}=\frac{s}{\mathrm{K}}.$$

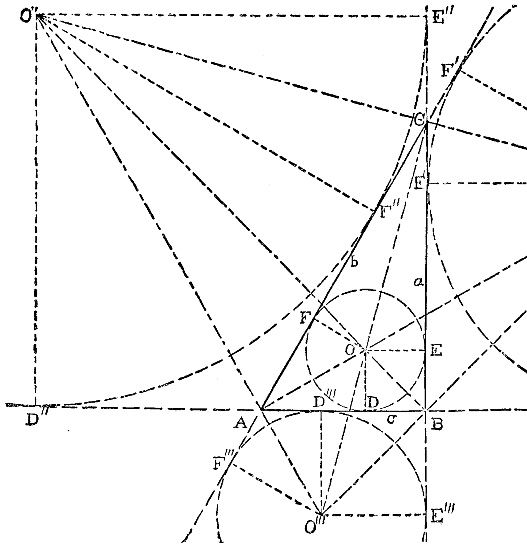
§ 5. INSCRIBED, ESCRIBED AND CIRCUMSCRIBED CIRCLES.

PROB. 4. TO FIND THE RADII OF THE CIRCLES INSCRIBED IN, ESCRIBED AND CIRCUMSCRIBED ABOUT, ANY TRIANGLE.

(1) *For the radius of the inscribed circle, divide the area by half the perimeter.*

(2) *For the radius of an escribed circle, divide the area by half the perimeter, less the side beyond which the circle lies.*

(3) *For the radius of the circumscribed circle, divide half of either side by the sine of the opposite angle.*



For, let ABC be any triangle, and let r stand for the radius of the inscribed circle, and r', r'', r''', for the radii of the escribed circles whose centers are O', O'', O''', respectively;
let R stand for the radius of the circumscribed circle; then:

(1) $\because\quad \text{K} = \frac{1}{2} r(a+b+c) = rs.$ [*geom.*

132] $\therefore\quad r = \frac{\text{K}}{s} = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$ Q. E. D.

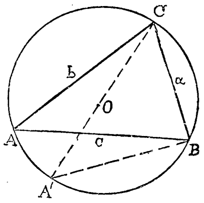
(2) $\because\quad \text{K} = \frac{1}{2} r'(-a+b+c) = r'(s-a),$

$= \frac{1}{2} r''(a-b+c) = r''(s-b),$

$= \frac{1}{2} r'''(a+b-c) = r'''(s-c);$ [*geom.*

133] $\therefore\quad r' = \frac{\text{K}}{s-a},\quad r'' = \frac{\text{K}}{s-b},\quad r''' = \frac{\text{K}}{s-c}.$ Q. E. D.

NOTE. The reader may prove the check-formulae



134] $\frac{1}{r} = \frac{1}{r'} + \frac{1}{r''} + \frac{1}{r'''}.$

135] $\text{K}^2 = r\,.\,r'\,.\,r''.\,r'''.$

(3) About ABC circumscribe a circle and draw CA′ a diameter; join A′B, then

$\because\quad \text{A} = \text{A}'$, and ∠ A′BC is a rt. ∠, [*geom.*

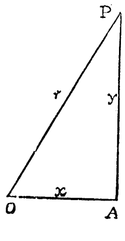
and $\because\quad \text{CA}' = \frac{a}{\sin \text{A}'} = \frac{a}{\sin \text{A}},$ [I. § 7.

136] $\therefore\quad \text{R} = \frac{\frac{1}{2}a}{\sin \text{A}}.$

So, $\text{R} = \frac{\frac{1}{2}b}{\sin \text{B}} = \frac{\frac{1}{2}c}{\sin \text{C}}.$ Q. E. D.

§ 6. EXERCISES.

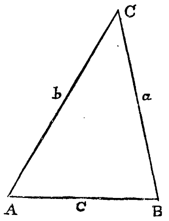
Solve the *right triangles* [both by use of natural functions and by use of logarithmic functions], given:



1. $r = 36.3$, $\text{O} = 50°$.
2. $x = 29.28$, $\text{P} = 37° 12'$.
3. $r = 125$, $y = 105$.
4. $x = 29.275$, $y = 39.07$.
5. $r = 37.09$, $y = .379$.
6. $r = 1311$, $\text{O} = 89° 18'$.

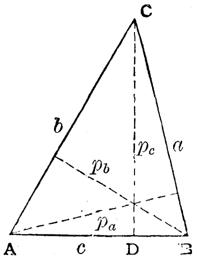
Solve the *oblique triangles* [both methods], given:

7. $a = 25.3$, $b = 136$, $C = 98° 15'$.
8. $A = 34°$, $B = 95°$, $c = 13.89$.
9. $a = 127$, $b = 64.9$, $c = 152.16$.
10. $a = 18$, $b = 20$, $A = 55° 24'$.
11. $a = 10$, $b = 20$, $A = 30°$.
12. $a = 16$, $b = 20$, $A = 86° 40'$.
13. $a = 20$, $b = 20$, $A = 47° 9'$.
14. $a = 24$, $b = 20$, $A = 37° 36'$.
15. $a = 24$, $b = 20$, $A = 120°$.
16. $a = 20$, $b = 20$, $A = 135°$.
17. $a = 16$, $b = 20$, $A = 150°$.
18. $a = 127$, $b = 254$, $c = 380$.
19. $a = 2000$, $b = 1999$, $A = 91°$.



Find the *areas* of the triangles, and the three perpendiculars, p_a, p_b, p_c, given:

20. $a = 12.5$, $b = 25$, $C = 36°$.
21. $A = 37° 18'$, $B = 92° 18'$, $c = 39.5$.
22. $a = 29.7$, $b = 6.238$, $c = 34.21$.

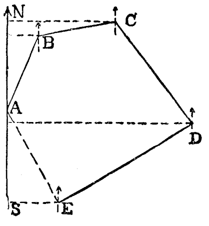


Find the radii of the inscribed, escribed and circumscribed circles, given:

23. $a = 12.7$, $b = 22.8$, $c = 33.9$.
24. $A = 64° 19' 8''$, $B = 100° 2' 27''$, $c = 51.25$.
25. $a = 136$, $b = 95.2$, $C = 11° 37'$.

26. In surveying, the *bearing* of a point is the angle which its direction makes with the north and south line through the point of observation; its *latitude* is its distance north or south, and its *departure* is its distance east or west, from the datum-point.

If a surveyor, starting from A,
run N. 22° 37′ E., 3.37 chs. to B;
thence N. 80° 24′ E., 3.81 chs. to C;
thence S. 41° 12′ E., 5.29 chs. to D;
thence S. 62° 45′ W., 6.22½ chs. to E;
find the latitude and departure respectively of B, C, D, E, from A; and find the bearing and distance of A from E.



27. Divide the field above given into triangles and trapezoids, by means of parallels of latitude (east and west lines) through the corners, and thence find its area.

28. At 120 feet distance from the foot of a steeple standing on a plane, the angle of elevation of the top is $60^\circ\ 30'$; find the height.

29. From the top of a rock 326 feet above the sea the angle of depression of a ship's hull is 24°; find the distance of the ship.

30. A ladder $29\frac{1}{2}$ feet long standing in the street just reaches a window 24 feet high on one side of the street, and 21 feet high on the other side; how wide is the street?

31. What is the dip of the horizon from the top of a mountain $2\frac{3}{4}$ miles high, the earth's mean radius being 3956 miles?

32. From the top of a mountain $1\frac{1}{2}$ miles high, the dip of the horizon is $1^\circ\,36'\,52''$; find the earth's diameter.

33. Given the earth's mean radius 3956 miles, and the angle which this radius subtends at the sun $8''.81$; find the distance of the earth from the sun.

Fig. 1.

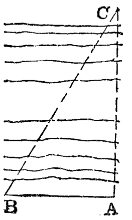


Fig. 2.

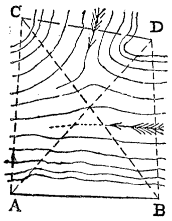
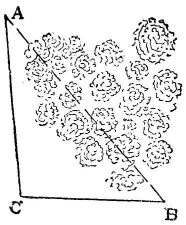


Fig. 3.



34. What is the distance across a river, when the base $AB = 475$ ft., $\angle A = 90^\circ$, $\angle B = 57^\circ\ 13'\ 20''$? (Fig. 1.)

35. What is the distance CD? (Fig. 2.) Given: the base $AB = 131\frac{1}{2}$ yds., $\angle BAD = 50^\circ$, $\angle BAC = 85^\circ\ 15'$, $\angle DBC = 38^\circ\,43'$, $\angle DBA = 94^\circ\,13'$. Prove the work by making two distinct computations from the data.

36. What is the distance AB? (Fig. 3.) Given: $CA = 131$ ft. 5 in., $BC = 109$ ft. 3 in., and $\angle C = 98^\circ\,34'$. Prove the work.

37. From the top of a hill I observe two milestones in the plain below, and in a straight line before me, and find their angles of depression 5° and 15°; what is the height of the hill?

38. Two observers on the same side of a balloon, and in the same vertical plane with it, are a mile apart, and find the angles of elevation 15° and 65° 30′ respectively; what is its height?

39. Two ships, lying half a mile apart, find the angles subtended by the other ship and a fort, respectively, 56° 19′ and 63° 14′; find the distance of each ship from the fort. Prove the work.

40. Bearings at sea are commonly reckoned in *points* and *quarter-points*, thus:

The points from north to east are: N., N. by E., N.N.E., N.E. by N., N.E., N.E. by E., E.N.E., E. by N., E.; and the quarter-points are: N., N. $\frac{1}{4}$ E., N. $\frac{1}{2}$ E., N. $\frac{3}{4}$ E., N. by E., N. by E. $\frac{1}{4}$ E., N. by E. $\frac{1}{2}$ E., N. by E. $\frac{3}{4}$ E., N.N.E.; N.E. by N. $\frac{3}{4}$ N., ..., N.E.; N.E. $\frac{1}{4}$ E, ..., E.N.E.; E. by N. $\frac{3}{4}$ N., ..., E.

Name in like manner the points and quarter-points from N. to W.; from S. to E.; from S. to W.

How many degrees in four points? in one point? in one quarter-point?

How many degrees and minutes from N.N.E. to E. by N.? to E. by S.? to S. by E.? to S. by E. $\frac{3}{4}$ E.?

41. A privateer lies 10 miles S.W. of a harbor, and observes a merchantman leave it in the direction E. by S., at the rate of 9 miles an hour; in what direction, and at what rate, must the privateer sail in order to overtake the merchantman in $1\frac{1}{2}$ hours?

42. From a vessel two headlands were observed: the first bore N.N.W., and the second N.E. by E.; then, sailing 12 miles E.N.E., the first bore N.W. and the second N.E. Find the bearing and distance of one headland from the other.

43. Find the ratio of the areas of two regular decagons, the one inscribed in, and the other circumscribed about, a circle.

44. Find the angle at which the side of a pyramid is inclined to the base, the sides being equilateral triangles and the base a square; thence find the diedral angle of a regular octaedron.

45. Find the diedral angle of an edge, the perpendicular from the vertex to the base, and the distance apart of two opposite edges, of a regular tetraedron whose edge is unity.

46. If ABC be any plane triangle, express $\sin\mathrm{A}$, $\cos\mathrm{A}$, $\tan\mathrm{A}$, and $\cos\mathrm{A}+\cos\mathrm{B}\cos\mathrm{C}$, in terms of the functions of B and C.

47. In any plane triangle ABC, prove that

$$a\cos\mathrm{B}+b\cos\mathrm{A}=c;$$

$$a\cos\mathrm{B}\cos\mathrm{C}+b\cos\mathrm{C}\cos\mathrm{A}+c\cos\mathrm{A}\cos\mathrm{B}$$

$$=a\sin\mathrm{B}\sin\mathrm{C}=b\sin\mathrm{C}\sin\mathrm{A}=c\sin\mathrm{A}\sin\mathrm{B}.$$ [Ex. 46

*48. From the known relations of the parts of a right triangle OAP, $\mathrm{O}+\mathrm{P}=90^\circ$, $r^2=x^2+y^2$, $x=r\cos\mathrm{O}$, $y=r\sin\mathrm{O}$, prove that

$$d\mathrm{P}=-d\mathrm{O},\qquad dr=\frac{x}{r}dx+\frac{y}{r}dy=\cos\mathrm{O}\,.\,dx+\sin\mathrm{O}\,.\,dy,$$

$$dx=\cos\mathrm{O}\,.\,dr-r\sin\mathrm{O}\,.\,d\mathrm{O},\qquad dy=\sin\mathrm{O}\,.\,dr+r\cos\mathrm{O}\,.\,d\mathrm{O},$$

and, by eliminating dr from the last two equations, that

$$d\mathrm{O}=-\frac{\sin\mathrm{O}}{r}\cdot dx+\frac{\cos\mathrm{O}}{r}\cdot dy=\frac{-ydx+xdy}{x^2+y^2},$$

wherein dx, dy, dr, $d\mathrm{O}$, $d\mathrm{P}$ stand for *total differentials* of x, y, r, O, P; *i.e.*, for any simultaneous infinitesimal changes in the quantities x, y, r, O, P, that are consistent with the known relations of the parts of a right triangle.

*49. If, in a right triangle, only the values of x and y be given, and if these have the *possible errors* $\pm x'$ and $\pm y'$ respectively; *i.e.*, if x may possibly differ from its assumed value by either $+x'$ or $-x'$, and y by either $+y'$ or $-y'$, these signs not being necessarily alike; show from Ex. 48 that the resulting values of r and O will have the possible errors

$$\pm\frac{xx'+yy'}{r},\ =\pm(x'\cos\mathrm{O}+y'\sin\mathrm{O}),$$

and $$\pm\frac{yx'+xy'}{r^2},\ =\pm\frac{1}{r}(x'\sin\mathrm{O}+y'\cos\mathrm{O});$$

wherein x' and y' are positive.

So, if only x and r be given, with the possible errors $\pm x'$ and $\pm r'$, find the possible errors of the other sides and angles.

So, if only x and O be given, or only r and O, with the possible errors $\pm x'$ and $\pm\mathrm{O}'$, or $\pm r'$ and $\pm\mathrm{O}'$.

*50. From the known relations of the parts of an oblique triangle ABC, $\mathrm{A} + \mathrm{B} + \mathrm{C} = 180°$, $a \sin \mathrm{B} = b \sin \mathrm{A}$, ..., prove that

a] $\quad d\mathrm{A} + d\mathrm{B} + d\mathrm{C} = 0,$

b] $\quad b \cos \mathrm{A} \,.\, d\mathrm{A} - a \cos \mathrm{B} \,.\, d\mathrm{B} - \sin \mathrm{B} \,.\, da + \sin \mathrm{A} \,.\, db = 0,$

$\quad c \cos \mathrm{B} \,.\, d\mathrm{B} - b \cos \mathrm{C} \,.\, d\mathrm{C} - \sin \mathrm{C} \,.\, db + \sin \mathrm{B} \,.\, dc = 0,$

$\quad a \cos \mathrm{C} \,.\, d\mathrm{C} - c \cos \mathrm{A} \,.\, d\mathrm{A} - \sin \mathrm{A} \,.\, dc + \sin \mathrm{C} \,.\, da = 0.$

From these, by elimination and reduction [104, Ex. 46], derive

c] $\quad b \,.\, d\mathrm{C} + c \cos \mathrm{A} \,.\, d\mathrm{B} - \sin \mathrm{A} \,.\, dc + \sin \mathrm{C} \,.\, da = 0,$

$\quad c \,.\, d\mathrm{B} + b \cos \mathrm{A} \,.\, d\mathrm{C} - \sin \mathrm{A} \,.\, db + \sin \mathrm{B} \,.\, da = 0,$

with four similar equations, which may be written from symmetry;

d] $\quad b \sin \mathrm{C} \,.\, d\mathrm{A} - da + \cos \mathrm{C} \,.\, db + \cos \mathrm{B} \,.\, dc = 0,$ [Ex. 47

with two similar equations, which may be written from symmetry.

*51. If in an oblique triangle only side a and angles B, C be given, and if their possible errors be $\pm \dfrac{a}{10\,000}$, $\pm 10''$, and $\pm 15''$, respectively, find the possible errors of A [Ex. 50, *a*]; of b [Ex. 50, *c*]; of c [Ex. 50, *c*].

Find the values of these possible errors when ABC is very nearly equilateral, 5000 feet on each side.

*52. Given the values of C, a, b, with the respective possible errors $\pm \mathrm{C}'$, $\pm a'$, $\pm b'$, deduce the possible errors of B and A [Ex. 50, *c*]; of c [Ex. 50, *d*].

*53. Given A, a, b, with possible errors $\pm \mathrm{A}'$, $\pm a'$, $\pm b'$, deduce the possible errors of B [Ex. 50, *b*]; of C and c.

*54. Given A, B, b, with possible errors $\pm \mathrm{A}'$, $\pm \mathrm{B}'$, $\pm b'$, find the possible errors of C, a and c; first, when, as in all the above cases, the computation is assumed to be exact; and secondly, when C, a, c are liable to the further possible errors $\pm \mathrm{C}''$, $\pm a''$, $\pm c''$ from omitted decimal-figures, etc., in the computation.

*55. Given a, b, c, with possible errors $\pm a'$, $\pm b'$, $\pm c'$, find the possible error of A; the possible error of computation being A''.

NOTE. If $\pm a'$, $\pm b'$, $\pm c'$, $\pm \mathrm{A}''$ denoted "probable errors," the probable error of A would be

$$\pm \sqrt{[(a''^2 + b''^2 \cos^2 \mathrm{C} + c''^2 \cos^2 \mathrm{B}) : b^2 \sin^2 \mathrm{C} + \mathrm{A}''^2]}.$$

In the same way, Ex. 49, 51–54 are adapted to probable errors.

SPHERICAL TRIGONOMETRY.

IV. SOLUTION OF SPHERICAL TRIANGLES.

§ 1. GEOMETRICAL PRINCIPLES.

If about the vertex of a triedral angle as center a sphere be described, the *traces* [intersections] of the three faces of the triedral upon the surface of the sphere, are arcs of great circles, and together they constitute a *spherical triangle*, whose sides measure the face-angles, and whose angles measure the diedral angles, of the triedral. SPHERICAL TRIGONOMETRY is therefore the trigonometry of the triedral, and it treats of the numerical relations of the six parts of the triedral, viz.: the three face-angles and the three diedrals.

As the three faces of the triedral, when produced, are indefinite planes, and divide all space into eight solid angles, so the sides of the spherical triangle, when produced, are great circles, and divide the surface of the sphere into eight triangles. Of these eight triangles, any two that are diametrically opposite are symmetrical, and any two not opposite have one side and its opposite angle the same in each, while the remaining sides and angles of the one are supplementary to the corresponding sides and angles of the other.

In this treatise only those triangles are considered whose parts are positive and each less than 180°; but in Astronomy the *general spherical triangle* is used, which is free from this restriction. In the restricted triangle above named the following principles hold true; they are proved in Geometry, and grouped together here for the convenience of the reader:

The sum of the three sides lies between 0° and 360°.

The sum of the three angles lies between 180° and 540°.

Each side is less than the sum of the other two.

Each angle is greater than the difference between 180° and the sum of the other two.

Of any two unequal sides, the greater lies opposite the greater angle; and conversely.

Each side or angle is the supplement of the corresponding angle or side of the polar triangle.

If two sides of a triangle are equal, so also are the opposite angles; and conversely.

The perpendicular from the vertex of an isosceles triangle to the base bisects both the vertical angle and the base.

Equilateral triangles, in general, are not similar.

A spherical triangle is, in general, determined when any three of its six parts are known.

The area is to the hemisphere as the excess of the sum of the angles over 180° is to 360°.

The following principles, proved later, are added:

In a right spherical triangle: An oblique angle and its opposite side are of the same species [both less than 90°, both greater than 90°, or both equal to 90°]. [Thm. 1 Cor. 1

If the hypothenuse is less than 90°, the other two sides are of the same species, and so are the two oblique angles; but if the hypothenuse is greater than 90° they are of different species, and so are the two oblique angles. [Thm. 1 Cor. 2

If the hypothenuse equals 90°, another side and its opposite angle are each equal to 90°; and if another side or its opposite angle equals 90°, the hypothenuse equals 90°. [Thm. 1 Cor. 3

No side is nearer to 90° than its opposite angle. [Thm. 1 Cor. 4

In any spherical triangle: A side that differs from 90° not less than another side is of the same species as its opposite angle. [Thm. 3 Cor.

An angle that differs from 90° not less than another angle is of the same species as its opposite side. [Thm. 4 Cor. 1

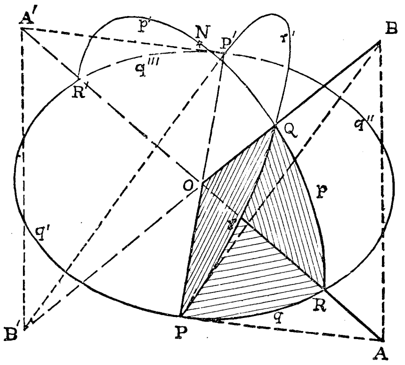
There are at least two sides which are of the same species as their opposite angles. [Thm. 4 Cor. 2

The half-sum of any two sides and the half-sum of their opposite angles are of the same species. [Thm. 12 Cor.

§ 2. NAPIER'S RULES FOR THE RIGHT TRIANGLE.

THEOREM 1. *In a right spherical triangle, if the right angle be ignored, and if, of the five remaining parts, the two sides be taken, and for the hypothenuse and the two oblique angles their complements be substituted, then the sine of any one of the five parts* (called the middle part) *equals:*

(1) *The product of the cosines of the two opposite parts;*
(2) *The product of the tangents of the two adjacent parts.*



[The figure shows four right spherical triangles:

PQR,
p, q and r all $< 90°$;
PQR′,
p' and $q' > 90°$, $r < 90°$;
P′QR,
$p < 90°$, q'' and $r' > 90°$;
P′QR′,
p' and $r' > 90°$, $q''' < 90°$.

The demonstration applies to all alike.]

Let PQR be any spherical triangle, wherein R is a right angle; P and Q oblique angles, either acute or obtuse; r the hypothenuse; p and q sides opposite P and Q respectively; then will:

151] $\sin p = \sin r \sin P$ and $= \tan q \cot Q$;
152] $\sin q = \sin r \sin Q$ and $= \tan p \cot P$;
153] $\cos r = \cos p \cos q$ and $= \cot P \cot Q$;
154] $\cos P = \cos p \sin Q$ and $= \tan q \cot r$;
155] $\cos Q = \cos q \sin P$ and $= \tan p \cot r$.

For, join the vertices P, Q, R to O, the center of the sphere, and through either oblique angle, as P, draw PA and PB, ⊥ OP, and meeting OR and OQ in A and B;
then also is AB ⊥ OA and PA, [*geom.*
and $\triangle$ OPA, OPB, OAB and PAB are right-angled at P, P, A and A.

But arcs p, q and r measure $\angle$s AOB, POA and POB, and $\mathrm{P} = \angle \mathrm{APB}$; [*geom.*

$$\therefore\ \sin p = \frac{\mathrm{AB}}{\mathrm{OB}},\quad \cos p = \frac{\mathrm{OA}}{\mathrm{OB}},\quad \tan p = \frac{\mathrm{AB}}{\mathrm{OA}};$$

$$\sin q = \frac{\mathrm{PA}}{\mathrm{OA}},\quad \cos q = \frac{\mathrm{OP}}{\mathrm{OA}},\quad \tan q = \frac{\mathrm{PA}}{\mathrm{OP}};$$

$$\sin r = \frac{\mathrm{PB}}{\mathrm{OB}},\quad \cos r = \frac{\mathrm{OP}}{\mathrm{OB}},\quad \tan r = \frac{\mathrm{PB}}{\mathrm{OP}};$$

$$\sin \mathrm{P} = \frac{\mathrm{AB}}{\mathrm{PB}},\quad \cos \mathrm{P} = \frac{\mathrm{PA}}{\mathrm{PB}},\quad \tan \mathrm{P} = \frac{\mathrm{AB}}{\mathrm{PA}}.$$ [I. § 7

$$\therefore\ \sin p, = \frac{\mathrm{AB}}{\mathrm{OB}}, = \frac{\mathrm{PB}}{\mathrm{OB}} \cdot \frac{\mathrm{AB}}{\mathrm{PB}} = \sin r \sin \mathrm{P};$$

$$\sin q, = \frac{\mathrm{PA}}{\mathrm{OA}}, = \frac{\mathrm{AB}}{\mathrm{OA}} \cdot \frac{\mathrm{PA}}{\mathrm{AB}} = \tan p \cot \mathrm{P};$$

$$\cos r, = \frac{\mathrm{OP}}{\mathrm{OB}}, = \frac{\mathrm{OA}}{\mathrm{OB}} \cdot \frac{\mathrm{OP}}{\mathrm{OA}} = \cos p \cos q;$$

$$\cos \mathrm{P}, = \frac{\mathrm{PA}}{\mathrm{PB}}, = \frac{\mathrm{PA}}{\mathrm{OP}} \cdot \frac{\mathrm{OP}}{\mathrm{PB}} = \tan q \cot r.$$ Q. E. D.

So, ∵ either oblique angle may be P and the other Q,

$$\therefore\ \sin q = \sin r \sin \mathrm{Q};$$

$$\sin p = \tan q \cot \mathrm{Q};$$

$$\cos \mathrm{Q} = \tan p \cot r.$$ Q. E. D.

And, ∵ $\cot \mathrm{P} = \dfrac{\sin q}{\tan p}$ and $\cot \mathrm{Q} = \dfrac{\sin p}{\tan q}$, [above

$$\therefore\ \cot \mathrm{P} \cot \mathrm{Q} = \frac{\sin p \sin q}{\tan p \tan q} = \cos p \cos q;$$ [35

but $\cos r = \cos p \cos q$, [above

$$\therefore\ \cos r = \cot \mathrm{P} \cot \mathrm{Q}.$$ Q. E. D

And, ∵ $\cos \mathrm{P} = \tan q \cot r$, $\cos p = \dfrac{\cos r}{\cos q}$, $\sin \mathrm{Q} = \dfrac{\sin q}{\sin r}$, [above

$$\therefore\ \cos \mathrm{P} = \cos p \sin \mathrm{Q}.$$

So, $\cos \mathrm{Q} = \cos q \sin \mathrm{P}$. Q. E. D.

Cor. 1. *In a right spherical triangle, an oblique angle and its opposite side are of the same species* [both less than 90°, both greater than 90°, or both equal to 90°].

For, $\because$ $\cos P = \cos p \sin Q$, [154

and $\because$ $\sin Q$ is not 0, and is always positive, [§ 1, I. Thm. 3

$\therefore$ $\cos P$ and $\cos p$ are both positive, or both negative, or both zero;

$\therefore$ P and p are both $< 90°$, or both $> 90°$, or both $= 90°$.

So, Q and q are ..... Q. E. D.

COR. 2. *In a right spherical triangle, if the hypothenuse is less than* 90°, *the other two sides are of the same species, and so are the two oblique angles; but, if the hypothenuse is greater than* 90°, *they are of different species, and so are the two oblique angles.*

For, $\because$ all parts of a triangle are positive and all less than 180°,

and $\because$ $\cos r = \cos p \cos q$, and $= \cot P \cot Q$, [153

$\therefore$ if $\cos r$ is positive, then $\cos p$ and $\cos q$ are both positive or both negative; and so are $\cot P$ and $\cot Q$;

i.e., if $r < 90°$, p and q are both $< 90°$, or both $> 90°$, and so are P and Q; Q. E. D. [I. Thm. 3

but, if $\cos r$ is negative, then $\cos p$ and $\cos q$ are one positive and the other negative; and so are $\cot P$ and $\cot Q$;

i.e., if $r > 90°$, p and q are one of them $< 90°$ and the other $> 90°$, and so are P and Q. Q. E. D. [I. Thm. 3

COR. 3. *In a right spherical triangle, if the hypothenuse equals* 90°, *another side and its opposite angle are also each equal to* 90°; *and conversely, if another side or its opposite angle equals* 90°, *the hypothenuse equals* 90°.

For, if $\cos r = 0$, then $\cos p \cos q = 0$, and $\cot P \cot Q = 0$, [153

$\therefore$ p or $q = 90°$, and P or $Q = 90°$, [I. Thm. 14

i.e., whichever of the sides and angles p, q, P or $Q = 90°$, the opposite angle or side $= 90°$. Q. E. D. [Cor. 1

So, if p or $P = 90°$, then $\cos p = 0$ or $\cot P = 0$,

$\therefore$ $\cos r = 0$, and $r = 90°$. [153

So, if q or $Q = 90°$, $r = 90°$. Q. E. D.

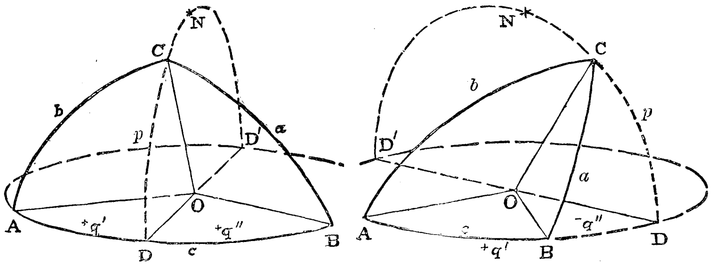
COR. 4. *In a right spherical triangle, no side is nearer to* 90° *than its opposite angle.*

For, if $p \sim 90° < P \sim 90°$,

then $\sin q, = \tan p \cot P, = \tan p : \tan P, > 1$; which is absurd.

§ 3. GENERAL PROPERTIES OF SPHERICAL TRIANGLES.

THM. 2. *In any spherical triangle the sines of the sides are proportional to the sines of the opposite angles.*



Let ABC be any spherical triangle; then will:

156] $\sin a : \sin b = \sin A : \sin B$;

$\sin b : \sin c = \sin B : \sin C$;

$\sin c : \sin a = \sin C : \sin A$.

For, draw CD, $= p$, $\perp$ AB; then:

$\because$ $\sin p = \sin a \sin B$, and $\sin p = \sin b \sin A$, [151

$\therefore$ $\sin a \sin B = \sin b \sin A$.

$\therefore$ $\sin a : \sin b = \sin A : \sin B$. [Thm. prop'n

So, $\sin b : \sin c = \sin B : \sin C$;

and $\sin c : \sin a = \sin C : \sin A$. Q. E. D.

NOTE. This theorem may be stated more symmetrically thus:

$$\sin a : \sin b : \sin c = \sin A : \sin B : \sin C;$$

or thus:

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}.$$

COR. *In any spherical triangle* ABC:

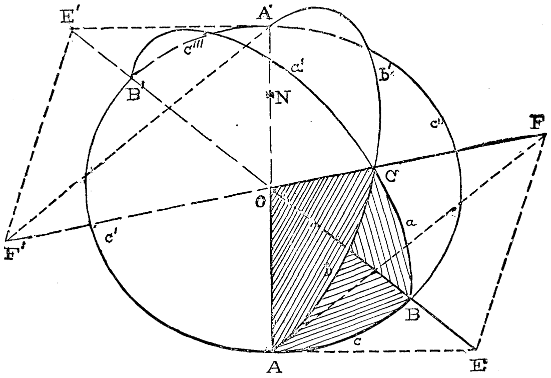
$$\sin a = \frac{\sin b \sin A}{\sin B} = \frac{\sin c \sin A}{\sin C}; \qquad \sin A = \frac{\sin B \sin a}{\sin b} = \frac{\sin C \sin a}{\sin c};$$

$$\sin b = \frac{\sin c \sin B}{\sin C} = \frac{\sin a \sin B}{\sin A}; \qquad \sin B = \frac{\sin C \sin b}{\sin c} = \frac{\sin A \sin b}{\sin a};$$

$$\sin c = \frac{\sin a \sin C}{\sin A} = \frac{\sin b \sin C}{\sin B}; \qquad \sin C = \frac{\sin A \sin c}{\sin a} = \frac{\sin B \sin c}{\sin b}.$$

THM. 3. *In any spherical triangle* ABC :

157]
$$\cos a = \cos b \cos c + \sin b \sin c \cos A,$$
$$\cos b = \cos c \cos a + \sin c \sin a \cos B,$$
$$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$



[The figure shows four oblique spherical triangles :

ABC, a, b and c all $< 90°$; AB'C, a' and $c' > 90°$, $b < 90°$;

A'BC, $a < 90°$, b' and $c'' > 90°$; A'B'C, a' and $b' > 90°$, $c''' < 90°$.

The demonstration applies to all alike.]

For, join the vertices ABC to O, the center of the sphere, and through A draw AE and AF, ⊥ OA, and meeting OB and OC in E and F ; then $\triangle$ OAE and OAF are right-angled at A.

But arcs a, b and c measure $\angle$s EOF, AOF and AOE,

and $A = \angle$ EAF ; then : [*geom.*

$$\because \quad \cos a = \frac{OE^2 + OF^2 - EF^2}{2\,OE\,.\,OF}, \qquad [108$$

and $\because \quad OE^2 = OA^2 + AE^2, \quad$ and $\quad OF^2 = OA^2 + AF^2$; [*geom.*

$$\therefore \quad \cos a = \frac{OA^2 + AE^2 + OA^2 + AF^2 - EF^2}{2\,OE\,.\,OF}$$

$$= \frac{OA^2}{OE\,.\,OF} + \frac{AE^2 + AF^2 - EF^2}{2\,OE\,.\,OF}$$

$$= \frac{OA}{OF} \cdot \frac{OA}{OE} + \frac{AF}{OF} \cdot \frac{AE}{OE} \cdot \frac{AE^2 + AF^2 - EF^2}{2\,AE\,.\,AF}.$$

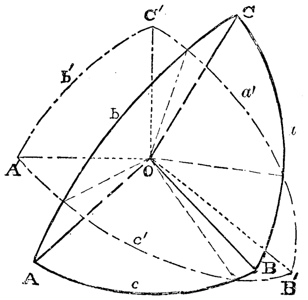
But
$$\cos A = \frac{AE^2 + AF^2 - EF^2}{2\,AE\,.\,AF}. \qquad [108$$

$\therefore\quad \cos a = \cos b \cos c + \sin b \sin c \cos \mathrm{A}.$ [I. § 7
So, $\quad \cos b = \cos c \cos a + \sin c \sin a \cos \mathrm{B},$
and $\quad \cos c = \cos a \cos b + \sin a \sin b \cos \mathrm{C}.$ Q. E. D.

COR. *In any spherical triangle, a side that differs from* 90° *not less than another side, is of the same species as its opposite angle.*

For, $\because\quad \cos \mathrm{A} = \dfrac{\cos a - \cos b \cos c}{\sin b \sin c},$ [157
and $\because$ the product $\sin b \sin c$ is always positive, [I. Thm. 3
$\therefore$ $\cos \mathrm{A}$ has the same sign as $(\cos a - \cos b \cos c)$.
But if a differs from 90° not less than b or c,
then $\cos a$ is as *large* [great numerically] as $\cos b$ or $\cos c$,
and if $\cos a = 0$, then $\cos b$ or $\cos c = 0$;
and $\because$ $\cos b$ and $\cos c$ are both *smaller* [less numerically] than 1,
$\therefore$ $\cos a$ is larger than $\cos b \cos c$;
$\therefore$ $\cos a$ gives sign to $(\cos a - \cos b \cos c)$;
$\therefore$ $\cos \mathrm{A}$ has the same sign as $\cos a$, [above
or if $\cos a = 0$, then $\cos \mathrm{A} = 0$;
i.e., both are positive, or both negative, or both zero;
$\therefore$ A and a, both $< 90°$, or both $> 90°$, or both $= 90°$. Q. E. D.

THM. 4. *In any spherical triangle* ABC:



158] $\cos \mathrm{A} = -\cos \mathrm{B} \cos \mathrm{C} + \sin \mathrm{B} \sin \mathrm{C} \cos a,$
$\cos \mathrm{B} = -\cos \mathrm{C} \cos \mathrm{A} + \sin \mathrm{C} \sin \mathrm{A} \cos b,$
$\cos \mathrm{C} = -\cos \mathrm{A} \cos \mathrm{B} + \sin \mathrm{A} \sin \mathrm{B} \cos c.$

For, let the triangle $\mathrm{A'B'C'}$ be polar to ABC;

then, $\because\quad a' = \pi - \mathrm{A},\quad b' = \pi - \mathrm{B},\quad c' = \pi - \mathrm{C},$ and $\mathrm{A'} = \pi - a,$ [*geom.*
and $\because\quad \cos a' = \cos b' \cos c' + \sin b' \sin c' \cos \mathrm{A'},$ [157
$\therefore -\cos \mathrm{A} = (-\cos \mathrm{B})(-\cos \mathrm{C}) + \sin \mathrm{B} \sin \mathrm{C}(-\cos a),$ [10, 11
$\therefore\quad \cos \mathrm{A} = -\cos \mathrm{B} \cos \mathrm{C} + \sin \mathrm{B} \sin \mathrm{C} \cos a.$
So, $\quad \cos \mathrm{B} = -\cos \mathrm{C} \cos \mathrm{A} + \sin \mathrm{C} \sin \mathrm{A} \cos b,$
and $\quad \cos \mathrm{C} = -\cos \mathrm{A} \cos \mathrm{B} + \sin \mathrm{A} \sin \mathrm{B} \cos c.$ Q. E. D.

COR. 1. *In any spherical triangle, an angle that differs from* 90° *not less than another angle, is of the same species as its opposite side.*

For, ∵ $\cos a = \dfrac{\cos A + \cos B \cos C}{\sin B \sin C}$, [158

and ∵ the product $\sin B \sin C$ is always positive, [I. Thm. 3

∴ $\cos a$ has the same sign as $(\cos A + \cos B \cos C)$.

But if A differs from 90° not less than B or C,

then $\cos A$ is at least as large as $\cos B$ or $\cos C$,

and if $\cos A = 0$, then $\cos B$ or $\cos C = 0$;

and ∵ $\cos B$ and $\cos C$ are both smaller than 1,

∴ $\cos A$ is larger than $\cos B \cos C$;

∴ $\cos A$ gives sign to $(\cos A + \cos B \cos C)$;

∴ $\cos a$ has the same sign as $\cos A$, [above

or if $\cos A = 0$ then $\cos a = 0$;

i.e., both are positive, or both negative, or both zero;

∴ a and A are both $< 90°$, or both $> 90°$, or both $= 90°$.

Q. E. D. [I. Thms. 3, 14

COR. 2. *In any spherical triangle there are at least two sides which are of the same species as their opposite angles.*

This is a direct consequence of Thm. 3 Cor. and Thm. 4 Cor. 1.

THM. 5. *In any spherical triangle* ABC:

159] $$\sin \tfrac{1}{2} A = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}},$$

$$\sin \tfrac{1}{2} B = \sqrt{\frac{\sin(s-c)\sin(s-a)}{\sin c \sin a}},$$

$$\sin \tfrac{1}{2} C = \sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a \sin b}}. \qquad [s = \tfrac{1}{2}(a+b+c)$$

For, ∵ $2\sin^2 \tfrac{1}{2} A = 1 - \cos A$ [63

$$= 1 - \frac{\cos a - \cos b \cos c}{\sin b \sin c} \qquad [157$$

$$= \frac{\cos b \cos c + \sin b \sin c - \cos a}{\sin b \sin c}$$

$$= \frac{\cos(b-c) - \cos a}{\sin b \sin c} \qquad [42$$

$$= \frac{-2\sin\frac{1}{2}(b-c+a)\sin\frac{1}{2}(b-c-a)}{\sin b\sin c} \qquad [71$$

$$= \frac{2\sin\frac{1}{2}(a-b+c)\sin\frac{1}{2}(a+b-c)}{\sin b\sin c} \qquad [1$$

$$= \frac{2\sin(s-b)\sin(s-c)}{\sin b\sin c},$$

$$\therefore\quad \sin\tfrac{1}{2}\mathrm{A} = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b\sin c}}.$$

So, $\sin\tfrac{1}{2}\mathrm{B} = \sqrt{\dfrac{\sin(s-c)\sin(s-a)}{\sin c\sin a}},$

and $\sin\tfrac{1}{2}\mathrm{C} = \sqrt{\dfrac{\sin(s-a)\sin(s-b)}{\sin a\sin b}}.$ Q. E. D.

THM. 6. *In any spherical triangle* ABC:

160] $\cos\tfrac{1}{2}\mathrm{A} = \sqrt{\dfrac{\sin s\sin(s-a)}{\sin b\sin c}},$

$$\cos\tfrac{1}{2}\mathrm{B} = \sqrt{\frac{\sin s\sin(s-b)}{\sin c\sin a}},$$

$$\cos\tfrac{1}{2}\mathrm{C} = \sqrt{\frac{\sin s\sin(s-c)}{\sin a\sin b}}. \qquad [s=\tfrac{1}{2}(a+b+c)$$

For, $\because\ 2\cos^2\tfrac{1}{2}\mathrm{A} = 1+\cos\mathrm{A}$ [64

$$= 1+\frac{\cos a-\cos b\cos c}{\sin b\sin c} \qquad [157$$

$$= \frac{\cos a-\cos b\cos c+\sin b\sin c}{\sin b\sin c}$$

$$= \frac{\cos a-\cos(b+c)}{\sin b\sin c} \qquad [41$$

$$= \frac{-2\sin\frac{1}{2}(a+b+c)\sin\frac{1}{2}(a-b-c)}{\sin b\sin c} \qquad [71$$

$$= \frac{2\sin\frac{1}{2}(a+b+c)\sin\frac{1}{2}(-a+b+c)}{\sin b\sin c} \qquad [1$$

$$= \frac{2\sin s\sin(s-a)}{\sin b\sin c},$$

$$\therefore\quad \cos\tfrac{1}{2}\mathrm{A} = \sqrt{\frac{\sin s\sin(s-a)}{\sin b\sin c}}.$$

So, $$\cos\tfrac{1}{2}\mathrm{B} = \sqrt{\frac{\sin s \sin(s-b)}{\sin c \sin a}},$$

and $$\cos\tfrac{1}{2}\mathrm{C} = \sqrt{\frac{\sin s \sin(s-c)}{\sin a \sin b}}.$$ Q. E. D.

THM. 7. *In any spherical triangle* **ABC**:

161] $$\tan\tfrac{1}{2}\mathrm{A} = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)}},$$

$$\tan\tfrac{1}{2}\mathrm{B} = \sqrt{\frac{\sin(s-c)\sin(s-a)}{\sin s \sin(s-b)}},$$

$$\tan\tfrac{1}{2}\mathrm{C} = \sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin s \sin(s-c)}}. \qquad [s=\tfrac{1}{2}(a+b+c)$$

The reader may prove the theorem by dividing the value of $\sin\frac{1}{2}\mathrm{A}$ by that of $\cos\frac{1}{2}\mathrm{A}$ [Thms. 5, 6]; and so for $\tan\frac{1}{2}\mathrm{B}$ and $\tan\frac{1}{2}\mathrm{C}$.

THM. 8. *In any spherical triangle* **ABC**:

162] $$\sin\tfrac{1}{2}a = \sqrt{\frac{-\cos \mathrm{S} \cos(\mathrm{S}-\mathrm{A})}{\sin \mathrm{B} \sin \mathrm{C}}},$$

$$\sin\tfrac{1}{2}b = \sqrt{\frac{-\cos \mathrm{S} \cos(\mathrm{S}-\mathrm{B})}{\sin \mathrm{C} \sin \mathrm{A}}},$$

$$\sin\tfrac{1}{2}c = \sqrt{\frac{-\cos \mathrm{S} \cos(\mathrm{S}-\mathrm{C})}{\sin \mathrm{A} \sin \mathrm{B}}}. \qquad [\mathrm{S}=\tfrac{1}{2}(\mathrm{A}+\mathrm{B}+\mathrm{C})$$

For, let the triangle $\mathrm{A'B'C'}$ be polar to ABC:

then ∵ $$\left.\begin{array}{lll} a'=\pi-\mathrm{A}, & b'=\pi-\mathrm{B}, & c'=\pi-\mathrm{C}, \\ \mathrm{A}'=\pi-a, & \mathrm{B}'=\pi-b, & \mathrm{C}'=\pi-c, \end{array}\right\} \qquad [\textit{geom.}$$

and ∵ $$\cos\tfrac{1}{2}\mathrm{A}' = \sqrt{\frac{\sin s' \sin(s'-a')}{\sin b' \sin c'}} \qquad [160$$

$$= \sqrt{\frac{\sin(\frac{1}{2}a'+\frac{1}{2}b'+\frac{1}{2}c')\sin(-\frac{1}{2}a'+\frac{1}{2}b'+\frac{1}{2}c')}{\sin b' \sin c'}},$$

$$\therefore \cos\tfrac{1}{2}(\pi-a) = \sqrt{\frac{\sin[\frac{3\pi}{2}-(\frac{1}{2}\mathrm{A}+\frac{1}{2}\mathrm{B}+\frac{1}{2}\mathrm{C})]\sin[\frac{\pi}{2}-(-\frac{1}{2}\mathrm{A}+\frac{1}{2}\mathrm{B}+\frac{1}{2}\mathrm{C})]}{\sin(\pi-\mathrm{B})\sin(\pi-\mathrm{C})}},$$

$$\therefore \quad \sin\tfrac{1}{2}a = \sqrt{\frac{-\cos \mathrm{S} \cos(\mathrm{S}-\mathrm{A})}{\sin \mathrm{B} \sin \mathrm{C}}}. \qquad [5, 16, 4, 10$$

So, $$\sin\tfrac{1}{2}b=\sqrt{\frac{-\cos \mathrm{s}\cos(\mathrm{s}-\mathrm{B})}{\sin \mathrm{C}\sin \mathrm{A}}},$$

and $$\sin\tfrac{1}{2}c=\sqrt{\frac{-\cos \mathrm{s}\cos(\mathrm{s}-\mathrm{C})}{\sin \mathrm{A}\sin \mathrm{B}}}.$$ Q. E. D.

The reader may also prove the theorem directly from the formulae of Thm. 4:

$$\cos \mathrm{A}=-\cos \mathrm{B}\cos \mathrm{C}+\sin \mathrm{B}\sin \mathrm{C}\cos a$$

in the same manner as Thm. 5 was proved from the formulae of Thm. 3.

THM. 9. *In any spherical triangle* ABC:

163] $$\cos\tfrac{1}{2}a=\sqrt{\frac{\cos(\mathrm{s}-\mathrm{B})\cos(\mathrm{s}-\mathrm{C})}{\sin \mathrm{B}\sin \mathrm{C}}},$$
$$\cos\tfrac{1}{2}b=\sqrt{\frac{\cos(\mathrm{s}-\mathrm{C})\cos(\mathrm{s}-\mathrm{A})}{\sin \mathrm{C}\sin \mathrm{A}}},$$
$$\cos\tfrac{1}{2}c=\sqrt{\frac{\cos(\mathrm{s}-\mathrm{A})\cos(\mathrm{s}-\mathrm{B})}{\sin \mathrm{A}\sin \mathrm{B}}}. \qquad [\mathrm{s}=\tfrac{1}{2}(\mathrm{A}+\mathrm{B}+\mathrm{C})$$

The reader may prove the theorem directly from the formulae in Thm. 4, or prove it by aid of the polar triangle in the same manner as Thm. 8 was proved.

THM. 10. *In any spherical triangle* ABC:

164] $$\tan\tfrac{1}{2}a=\sqrt{\frac{-\cos \mathrm{s}\cos(\mathrm{s}-\mathrm{A})}{\cos(\mathrm{s}-\mathrm{B})\cos(\mathrm{s}-\mathrm{C})}},$$
$$\tan\tfrac{1}{2}b=\sqrt{\frac{-\cos \mathrm{s}\cos(\mathrm{s}-\mathrm{B})}{\cos(\mathrm{s}-\mathrm{C})\cos(\mathrm{s}-\mathrm{A})}},$$
$$\tan\tfrac{1}{2}c=\sqrt{\frac{-\cos \mathrm{s}\cos(\mathrm{s}-\mathrm{C})}{\cos(\mathrm{s}-\mathrm{A})\cos(\mathrm{s}-\mathrm{B})}}. \qquad [\mathrm{s}=\tfrac{1}{2}(\mathrm{A}+\mathrm{B}+\mathrm{C})$$

The reader may prove the theorem by dividing the value of $\sin\frac{1}{2}a$ by that of $\cos\frac{1}{2}a$ [Thms. 8, 9]; and so for $\tan\frac{1}{2}b$ and $\tan\frac{1}{2}c$.

NOTE. The radicals in [159–164] are all positive; for they are the sines, cosines and tangents of angles $\frac{1}{2}\mathrm{A}$, ….. $\frac{1}{2}a$, ….., which are all in the first quadrant.

THM. 11. *In any spherical triangle* ABC:

165] $\sin\frac{1}{2}(A+B)=\dfrac{\cos\frac{1}{2}(a\sim b)}{\cos\frac{1}{2}c}\cos\frac{1}{2}C;$

166] $\sin\frac{1}{2}(A\sim B)=\dfrac{\sin\frac{1}{2}(a\sim b)}{\sin\frac{1}{2}c}\cos\frac{1}{2}C;$

167] $\cos\frac{1}{2}(A+B)=\dfrac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}c}\sin\frac{1}{2}C;$

168] $\cos\frac{1}{2}(A\sim B)=\dfrac{\sin\frac{1}{2}(a+b)}{\sin\frac{1}{2}c}\sin\frac{1}{2}C.$

For, ∵ $\sin\frac{1}{2}(A\pm B)=\sin(\frac{1}{2}A\pm\frac{1}{2}B)$

$=\sin\frac{1}{2}A\cos\frac{1}{2}B\pm\cos\frac{1}{2}A\sin\frac{1}{2}B$ [39, 40

$=\dfrac{\sin(s-b)}{\sin c}\sqrt{\dfrac{\sin s\sin(s-c)}{\sin a\sin b}}\pm\dfrac{\sin(s-a)}{\sin c}\sqrt{\dfrac{\sin s\sin(s-c)}{\sin a\sin b}}$ [159,160

$=\dfrac{\sin(s-b)\pm\sin(s-a)}{\sin c}\cos\frac{1}{2}C$ [160

$$=\left\{\begin{array}{ll}\dfrac{2\sin\frac{1}{2}c\cos\frac{1}{2}(a\sim b)}{2\sin\frac{1}{2}c\cos\frac{1}{2}c}\cos\frac{1}{2}C & \text{[68, 60}\\[2ex] \dfrac{2\cos\frac{1}{2}c\sin\frac{1}{2}(a-b)}{2\sin\frac{1}{2}c\cos\frac{1}{2}c}\cos\frac{1}{2}C, & \text{[69, 60}\end{array}\right.$$

∴ $\sin\frac{1}{2}(A+B)=\dfrac{\cos\frac{1}{2}(a\sim b)}{\cos\frac{1}{2}c}\cos\frac{1}{2}C,$

and $\sin\frac{1}{2}(A\sim B)=\dfrac{\sin\frac{1}{2}(a\sim b)}{\sin\frac{1}{2}c}\cos\frac{1}{2}C.$ Q. E. D.

So, ∵ $\cos\frac{1}{2}(A\pm B)=\cos\frac{1}{2}A\cos\frac{1}{2}B\mp\sin\frac{1}{2}A\sin\frac{1}{2}B$ [41, 42

$=\dfrac{\sin s}{\sin c}\sqrt{\dfrac{\sin(s-a)\sin(s-b)}{\sin a\sin b}}\mp\dfrac{\sin(s-c)}{\sin c}\sqrt{\dfrac{\sin(s-a)\sin(s-b)}{\sin a\sin b}}$

$=\dfrac{\sin s\mp\sin(s-c)}{\sin c}\sin\frac{1}{2}C$ [159

$$=\left\{\begin{array}{ll}\dfrac{2\cos\frac{1}{2}(a+b)\sin\frac{1}{2}c}{2\sin\frac{1}{2}c\cos\frac{1}{2}c}\sin\frac{1}{2}C & \text{[69, 60}\\[2ex] \dfrac{2\sin\frac{1}{2}(a+b)\cos\frac{1}{2}c}{2\sin\frac{1}{2}c\cos\frac{1}{2}c}\sin\frac{1}{2}C, & \text{[68, 60}\end{array}\right.$$

∴ $\cos\frac{1}{2}(A+B)=\dfrac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}c}\sin\frac{1}{2}C,$

and $\cos\frac{1}{2}(A\sim B)=\dfrac{\sin\frac{1}{2}(a+b)}{\sin\frac{1}{2}c}\sin\frac{1}{2}C.$ Q. E. D.

These four formulae, with the like formulae found when the other sides and angles are employed, are called *Delambre's Formulae.*

Thm. 12. *In any spherical triangle* ABC:

169] $$\tan\tfrac{1}{2}(A+B)=\frac{\cos\frac{1}{2}(a\sim b)}{\cos\frac{1}{2}(a+b)}\cot\tfrac{1}{2}C;$$

170] $$\tan\tfrac{1}{2}(A\sim B)=\frac{\sin\frac{1}{2}(a\sim b)}{\sin\frac{1}{2}(a+b)}\cot\tfrac{1}{2}C;$$

171] $$\tan\tfrac{1}{2}(a+b)=\frac{\cos\frac{1}{2}(A\sim B)}{\cos\frac{1}{2}(A+B)}\tan\tfrac{1}{2}c;$$

172] $$\tan\tfrac{1}{2}(a\sim b)=\frac{\sin\frac{1}{2}(A\sim B)}{\sin\frac{1}{2}(A+B)}\tan\tfrac{1}{2}c.$$

The reader may prove the theorem by dividing the formulae of Thm. 11 one by another, viz.:

[165] by [167]; [166] by [168];
[168] by [167]; [166] by [165].

These four formulae, with the like formulae found when the other sides and angles are employed, are called *Napier's Analogies.*

Cor. *In any spherical triangle the sum of any two sides is less than, equal to, or greater than,* 180°, *according as the sum of the opposite angles is less than, equal to, or greater than,* 180°.

For, ∵ $\tan\frac{1}{2}(a+b)\cos\frac{1}{2}(A+B)=\tan\frac{1}{2}c\cos\frac{1}{2}(A\sim B)$, [171
and ∵ $\frac{1}{2}c$ and $\frac{1}{2}(A\sim B)$ are both less than 90°, and $\frac{1}{2}c$ is not 0,
whence the product $\tan\frac{1}{2}c\cos\frac{1}{2}(A\sim B)$ is positive, and not 0;
∴ the product $\tan\frac{1}{2}(a+b)\cos\frac{1}{2}(A+B)$ is positive, not 0,
∴ $\tan\frac{1}{2}(a+b)$ and $\cos\frac{1}{2}(A+B)$ are both positive or both negative, or one is ∞ and the other 0;
∴ $\frac{1}{2}(a+b)$ and $\frac{1}{2}(A+B)$ are both $<90°$, both $=90°$, or both $>90°$; [I. Thm. 3
∴ $(a+b)$ and $(A+B)$ are both $<180°$, both $=180°$, or both $>180°$. Q. E. D.

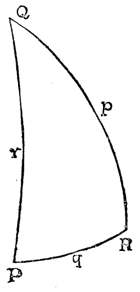
Note. Thm. 1 Cor. 1, and the Cors. to Thms. 3, 4 and 12 are summarized as follows:

The half-sum of any two sides and the half-sum of their opposite angles are of the same species; and so are either side and its opposite angle, unless the side be nearer to 90° than any other side, and the angle be nearer to 90° than any other angle.

§ 4. SOLUTION OF RIGHT TRIANGLES.

PROB. 1. TO SOLVE A RIGHT SPHERICAL TRIANGLE.

Let PQR be any spherical triangle right-angled at R; then the formulae of Thm. 1 (Napier's rules) apply directly, and with the right angle and any other two parts given, the remaining parts can be found. The computer will form equations containing three parts, two known and one unknown, and then solve these equations for the unknown part. He will observe that:



If the three parts are contiguous to each other, then

that which lies between the other two is the middle part, and the others are adjacent parts;

but if two parts lie together and the other apart from them, then

that which lies apart from the other two is the middle part, and the others are opposite parts.

The following rule will solve all cases:

Take each of the two given parts in turn for middle part, and apply that one of Napier's rules which brings in the other given part.

Take the remaining part for middle part, and apply that one of Napier's rules which brings in both of the parts just found.

For a check make the part last found the middle part, and apply that one of Napier's rules which brings in both the given parts.

The whole work requires but nine logarithms, or seven without the check, since two of the logarithmic functions are used twice over. The check is to be applied to the sine of the part last found. If the two values got for this sine, natural or logarithmic, differ by not more than three units in the last decimal place, the work is probably right, since the defects of the tables permit this discrepancy in the two results. If such discrepancy exist the mean of the two values may be used.

Case 1. *Given* p *and* q, *the two sides about the right angle*, then:

$$\begin{aligned} &\sin q = \tan p \cot \mathrm{P}, \quad \therefore \quad \cot \mathrm{P} = \cot p \sin q\,;\\ &\sin p = \tan q \cot \mathrm{Q}, \quad \therefore \quad \cot \mathrm{Q} = \sin p \cot q\,;\\ &\cos r = \cot \mathrm{P} \cot \mathrm{Q}\,; \ \textit{check} \ \cos r = \cos p \cos q. \end{aligned}$$

Note. One triangle is always possible, and but one; the parts q, P and Q are determined without ambiguity by the formulae.

Case 2. *Given the hypothenuse and one side*, for example p, then:

$$\begin{aligned} &\cos r = \cos p \cos q, \quad \therefore \quad \cos q = \sec p \cos r\,;\\ &\sin p = \sin r \sin \mathrm{P}, \quad \therefore \quad \sin \mathrm{P} = \sin p \csc r\,;\\ &\cos \mathrm{Q} = \cos q \sin \mathrm{P}\,; \ \textit{check} \ \cos \mathrm{Q} = \tan p \cot r. \end{aligned}$$

Note. A triangle is possible only when r is nearer to 90° than p is, or when r and p are both 90°;
for then only can

$$\cos q, = \frac{\cos r}{\cos p}, < 1\,; \quad \text{or} \quad \cos \mathrm{Q}, = \frac{\tan p}{\tan r}, < 1.$$

The formula would give two values to P, an acute angle and an obtuse one, supplementary to each other;
but, since p and P are both of the same species, [Thm. 1 Cor. 1
only one value of P is admissible. The parts q, P and Q are therefore determined without ambiguity by the formulae, unless p and r are both 90°, when $\cos \mathrm{Q}$ and $\cos q$ become indeterminate, and $q = \mathrm{Q}$ and $\mathrm{P} = 90°$.

Case 3. *Given an oblique angle and the adjacent side*, for example P and q, then:

$$\begin{aligned} &\sin q = \tan p \cot \mathrm{P}, \quad \therefore \quad \tan p = \sin q \tan \mathrm{P}\,;\\ &\cos \mathrm{P} = \tan q \cot r, \quad \therefore \quad \cot r = \cot q \cos \mathrm{P}\,;\\ &\cos \mathrm{Q} = \tan p \cot r\,; \ \textit{check} \ \cos \mathrm{Q} = \cos q \sin \mathrm{P}. \end{aligned}$$

Note. One triangle is always possible, and but one; the parts p, r and Q are determined without ambiguity by the formulae.

Case 4. *Given an oblique angle and the opposite side*, for example P and p, then:

$$\begin{aligned} &\sin p = \sin r \sin \mathrm{P}, \quad \therefore \quad \sin r = \sin p \csc \mathrm{P}\,;\\ &\cos \mathrm{P} = \cos p \sin \mathrm{Q}, \quad \therefore \quad \sin \mathrm{Q} = \sec p \cos \mathrm{P}\,;\\ &\sin q = \sin r \sin \mathrm{Q}\,; \ \textit{check} \ \sin q = \tan p \cot \mathrm{P}. \end{aligned}$$

NOTE. If p and P are not of the same species, no triangle is possible, [Thm. 1 Cor. 1

nor if p is nearer 90° than P. [Thm. 1 Cor. 4

If p and P are equal, but not 90°,

then $\sin q$, $\sin r$, and $\sin Q$, all, $= 1$;

∴ q, r, and Q, all, $= 90°$, [I. Thm. 14

and the triangle is biquadrantal.

If p and P are both 90°,

then r is also 90°, and the triangle is biquadrantal, [*geom.*

and q and Q are indeterminate.

If P is nearer 90° than p,

then two triangles are always possible,

for ∵ q, r and Q are all determined from their sines,

and ∵ to every sine correspond two angles, supplements of each other, [10

∴ q, r and Q may each have two values;

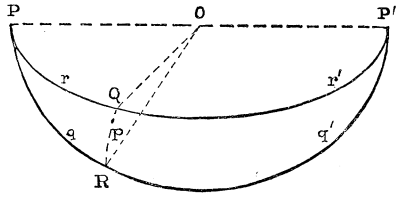
but, ∵ q and Q are of the same species, [Thm. 1 Cor. 1

∴ if r is less than 90°, q and Q are of the same species with p and P, and there is but one value for each of them. [Thm. 1 Cor. 2

So, if r is greater than 90°, q and Q are of the opposite species to p and P, and there is but one value for each of them;

∴ two triangles, and but two, are formed with the given data, viz.:

(1) that wherein $r < 90°$, and q and Q are of the same species with p and P.



(2) that wherein $r > 90°$, and q and Q are of the opposite species to p and P.

This ambiguity appears directly from the figure.

For, let PQR be a spherical triangle right-angled at R; produce the arcs PQ and PR to meet at P'; then:

∵ $P = P'$, [*geom.*

and ∵ ∠s PRQ and P′RQ are both right angles, [hypoth.

∴ two right triangles exist, PQR and P′QR, which have the same two parts given, p and P, and the remaining parts of the one triangle supplementary to those of the other.

CASE 5. *Given the hypothenuse and one oblique angle*, for example P, then:

$$\cos P = \tan q \cot r, \quad \therefore \quad \tan q = \tan r \cos P;$$
$$\cos r = \cot P \cot Q, \quad \therefore \quad \cot Q = \cos r \tan P;$$
$$\sin p = \tan q \cot Q; \quad \textit{check} \ \sin p = \sin r \sin P.$$

NOTE. One triangle is always possible, and but one. The part p is of the same species with P, and so can have but one of two possible values; the parts q and Q are determined without ambiguity by the formulae.

CASE 6. *Given the two oblique angles* P *and* Q, then:

$$\cos P = \cos p \sin Q, \quad \therefore \quad \cos p = \cos P \csc Q;$$
$$\cos Q = \cos q \sin P, \quad \therefore \quad \cos q = \csc P \cos Q;$$
$$\cos r = \cos p \cos q; \quad \textit{check} \ \cos r = \cot P \cot Q.$$

NOTE 1. The parts p, q and r are determined without ambiguity by the formulae; but the solution is possible only when $\cos P \csc Q$, $\csc P \cos Q$ and $\cot P \cot Q$ are each smaller than 1;

i.e., when $\cos P$ is smaller than $\sin Q$ and $\cos Q$ is smaller than $\sin P$,

and this when P is nearer to 90° than Q to 0° or 180°,

and when Q is nearer to 90° than P to 0° or 180°.

NOTE 2. The computer may follow a different order from that given; he may, at pleasure, find all the required parts directly from the given parts, or compute any one of the required parts, and then use that part in the computation of other parts. If, however, he make an error in the first part, that error is repeated and perhaps magnified, in the computation of all other parts which depend upon it; hence the importance of testing the results.

NOTE 3. Unless he use the special tables noted under the right plane triangle, the computer must avoid angles near 0°, 90° or

180°. For this purpose he may prove and use the following formulae:

173] $\sin^2 \frac{1}{2} r = \sin^2 \frac{1}{2} p \cos^2 \frac{1}{2} q + \cos^2 \frac{1}{2} p \sin^2 \frac{1}{2} q$; [153, 61; 36

174] $\tan^2 \frac{1}{2} p = \tan \frac{1}{2}(r + q) \tan \frac{1}{2}(r - q)$; [65, 153, 77

175] $\tan^2 \frac{1}{2} \mathrm{P} = \sin(r - q) : \sin(r + q)$; [65, 154, 57

176] $\tan \frac{1}{2} \mathrm{P} = \sin(r - q) : \sin p \cos q$; $= \frac{\sin p \cos q}{\sin(r+q)}$; [65, 154, 151

177] $\sin(p - q) = \sin p \tan \frac{1}{2} \mathrm{P} - \sin q \tan \frac{1}{2} \mathrm{Q}$; [176, 40, 153

178] $\tan^2 \frac{1}{2} p = \tan \frac{1}{2}(\mathrm{Q} + \mathrm{P} - 90°) : \tan \frac{1}{2}(\mathrm{Q} - \mathrm{P} + 90°)$; [65, 154, 72

179] $\tan^2 \frac{1}{2} r = - \cos(\mathrm{P} + \mathrm{Q}) : \cos(\mathrm{P} - \mathrm{Q})$. [65, 153, 58

§ 5. SOLUTION OF QUADRANTAL TRIANGLES, AND ISOSCELES TRIANGLES.

PROB. 2. TO SOLVE A QUADRANTAL TRIANGLE:

Find the triangle which is polar to the given triangle; it is a right triangle; solve it, and take the supplements of the parts thus found for the corresponding parts of the given triangle.

NOTE 1. Napier's rules apply directly to the quadrantal triangle, if the quadrant is ignored, and for the five parts are taken the two angles adjacent to the quadrant and the complements of the opposite angle and the two oblique sides.

NOTE 2. Manifestly, the biquadrantal triangle cannot be solved unless either the base or the vertical angle is given; for the remaining parts, two right angles and two quadrants, are quite independent of these two.

PROB. 3. TO SOLVE AN ISOSCELES TRIANGLE:

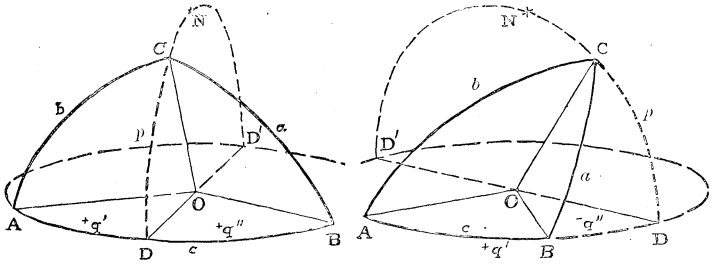
Draw an arc from the vertex to the middle of the base, thereby dividing the given triangle into two equal right triangles; solve one of these triangles.

NOTE. When only the base and the vertical angle are given, there are two triangles, one triangle or none, according as the base <, = or > the vertical angle. When only the two equal sides or the two equal angles are given, there is an infinite number of triangles. Otherwise, subject to the conditions in § 1, there is one triangle, and but one.

§ 6. SOLUTION OF OBLIQUE TRIANGLES.

PROB. 4. TO SOLVE AN OBLIQUE TRIANGLE.

FIRST METHOD. *By means of right triangles.*



Let ABC be any oblique spherical triangle, and a, b, c the sides opposite the vertices A, B, C respectively. Let N be the pole of AB, and through N and C draw a great circle, meeting the great circle AB at D and D', whereof D stands less than 180° in the direction AB from A; then is DC ⊥ AB. [*geom.*

Let p stand for DC; q', always positive, for AD; q'' for DB; C', always positive, for ∠ ACD; C'' for ∠ DCB.

CASE 1. *Given two sides and the included angle*, for example b, c and A; then:

In rt. △ ACD, b and A are known, whence p, q' and C' are found;

$q'' = c - q'$;

In rt. △ BCD, p and q'' are known, whence a, ∠ CBD and C'' are found;

$\mathrm{C} = \mathrm{C}' + \mathrm{C}''$.

NOTE 1. C'' is positive or negative, and $\mathrm{B} = \mathrm{CBD}$ or $180° - \mathrm{CBD}$, according as q'' is positive or negative with reference to AB, *i.e.*, according as $c >$ or $< q'$.

NOTE 2. There is always one triangle, and but one.

The parts are determined without ambiguity by the formulae.

CASE 2. *Given two angles and the included side*, for example C, A and b; then:

In rt. △ ACD, b and A are known, whence p, q' and C' are found;

$C'' = C - C'$;

In rt. △ BCD, p and C'' are known, whence a, q'' and ∠ CBD are found;

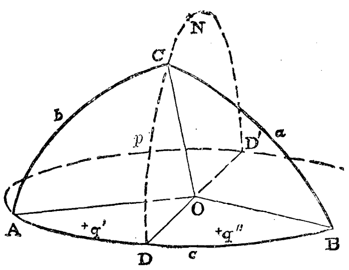
$c = q' + q''$.

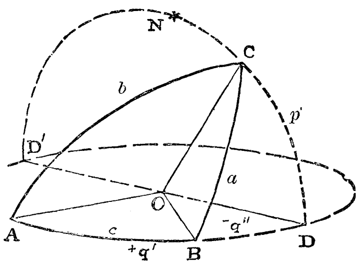
NOTE 1. q'' is positive or negative, and $B = CBD$ or $180° - CBD$, according as C'' is positive or negative with reference to positive rotation, *i.e.*, according as $C >$ or $< C'$.

NOTE 2. There is always one triangle, and but one.

The parts are determined without ambiguity from the formulae.

NOTE 3. If for the given triangle its polar be substituted, then two sides and the included angle are known, and Case 2 is embraced in Case 1. The required parts are the supplements of the parts found in the polar.





CASE 3. *Given two sides and an angle opposite one of them*, for example a, b and A; then:

In rt. △ ACD, b and A are known, whence p, q' and C' are found;

In rt. △ BCD, p and a are known, whence q'', ∠ CBD and C'' are found;

$c = q' + q''$;

$C = C' + C''$.

NOTE 1. q' is fully determined from the data, but q'' (found from its cosine) may be positive or negative; and there are two triangles, one triangle or none, according as $q'-q''$ and $q'+q''$, both, one of them or neither, lie between 0° and 180°.

The perpendicular CD falls within or without the triangle, and $B = CBD$ or $180° - CBD$, according as q'' is taken positive or negative with reference to the direction AB.

And, $\because$ A and $\angle$ CBD are always of the same species with p, [Thm. 1 Cor. 1

$\therefore$ A and B are of the same species or different species according as $B = CBD$ or $180° - CBD$;

i.e., according as q'' is positive or negative.

*NOTE 2. Whether there are two triangles, one triangle or none, may, in general, be known by inspection of the given parts:

(1) If a lies between b and its supplement, there is one triangle, and but one.

(2) If a equals b or its supplement, then:
If A is of the same species with a, and is not 90°, there is one triangle;
If A is of different species from a, there is no triangle;
If A, a and b are all 90°, there is an infinite number of triangles.

(3) If b lies between a and its supplement, and if a and A are of the same species, there are two triangles, one triangle or none, according as a is nearer 90° than p, as near it, or more remote from it; and there is no triangle if a and A are of different species.

For $\because$ $\triangle$BCD is possible if a is as near 90° as p, and of the same species;

$\therefore$ it is possible if a is as near 90° as b, and of the same species. [Prob. 1, Case 2, note

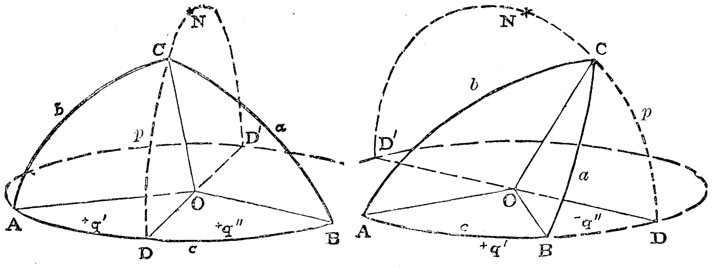
And $\because$ $\cos a = \cos p \cos q''$ and $\cos b = \cos p \cos q'$, [153

$\therefore$ $\cos a : \cos b = \cos q'' : \cos q'$, [Thm. prop'n

a proportion wherein a, b and q' are known, and q'' may, in general, be either positive or negative.

(1) ∴ If a is nearer 90° than b,
then q'' is nearer 90° than q', [I. § 23, Note 2
and of $q'+q''$ and $q'-q''$, one always, but never both, lies between 0° and 180°; the first or the second of them, according as $q' <$ or $> 90°$.

(2) If $a=b$, then $q''=q'$, [2
and if $a=180°-b$, then $q''=180°-q'$ or $-q''=180°-q'$; [11
∴ in either case, in general, of $q'+q''$ and $q'-q''$, one lies between 0° and 180°, and the other $=0°$ or 180°, and there is one triangle.
But if a and A are not of the same species,
then ∵ a differs from 90° not less than b,
∴ there is no triangle. [Thm. 3 Cor.
So, if a, b and A are all 90°,
then B is also 90°, and the triangle is biquadrantal and indeterminate. [Thm. 1 Cor. 3



(3) If b is nearer 90° than a,
then, if there is a triangle ACD, q' is nearer 90° than q'',
and, in general, $q'+q''$ and $q'-q''$ both lie between 0° and 180°.
But, if $p=a$,
then $q''=0$, and $q'+q''=q'-q''$,
∴ there is but one (a right) triangle.
So, if p is nearer 90° than a,
then $\cos q''>1$, which is impossible;
∴ there is no triangle ACD, and no triangle ABC.
And ∵ b is nearer 90° than a, [hypoth.
∴ there is no triangle when a and A are of different species. [Thm. 3 Cor.

CASE 4. *Given two angles and a side opposite one of them*, for example A, B and a.

In rt. △ BCD, a and ∠ CBD are known, whence p, q'' and C$''$ are found;
In rt. △ ACD, p and A are known, whence b, q' and C$'$ are found;
$c = q' + q''$;
$\text{C} = \text{C}' + \text{C}''$.

NOTE 1. C$''$ is fully determined by the data, but C$'$ (found from its sine) may be less or greater than 90°; [Prob. 1, Case 4, note and there are two triangles, one triangle or none, according as $\text{C}' - \text{C}''$ and $\text{C}' + \text{C}''$, both, one of them or neither, lie between 0° and 180°.

The perpendicular CD falls within or without the triangle, and C$''$ is positive or negative with reference to positive rotation, according as p and B are of the same or different species; [Thm. 1 Cor. 1 and b is less or greater than 90° according as A and C$'$ are of the same or different species. [Thm. 1 Cor. 2

*NOTE 2. Whether there are two triangles, one triangle, or none, may, in general, be known by inspection of the given parts:

(1) If A lies between B and its supplement, there is one triangle, and but one.

(2) If A equals B or its supplement, then:

If a is of the same species with A, and is not 90°, there is one triangle;
If a is of different species from A, there is no triangle;
If a, A and B are all 90°, there is an infinite number of triangles.

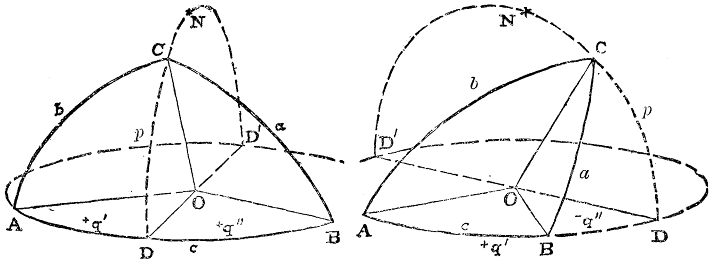
(3) If B lies between A and its supplement, and if A and a are of the same species, there are two triangles, one triangle, or none, according as A is nearer 90° than p, as near it, or more remote from it; and there is no triangle if A and a are of different species.

For ∵ ACD is possible if A is as near 90° as p, and of the same species;
∴ it is possible if A is as near 90° as B, and of the same species. [Prob. 1, Case 4, note

And ∵ $\cos A = \cos p \sin c'$ and $\cos CBD = \cos p \sin c''$, [154
∴ $\cos A : \cos CBD = \sin c' : \sin c''$, [Thm. prop'n
a proportion wherein A, CBD and c'' are known, and c' may, in general, have two supplementary values, c_1 and c_2, both positive. [10

(1) ∴ If A is nearer 90° than B,
then c'' is nearer 90° than c_1 and c_2, [§ 23, Note 2
and of $c_1 + c''$ and $c_2 + c''$, one always, but never both, lies between 0° and 180°.

(2) If $A = B$, then $c_1 = c''$ or $c_2 = c''$,
and if $A = 180° - B$, then $c_1 = - c''$ or $c_2 = - c''$;
∴ in either case, in general, of $c_1 + c''$ and $c_2 + c''$, one lies between 0° and 180°, and the other = 0° or 180°, and there is one triangle.
But, if A and *a* are not of the same species,
then ∵ A differs from 90° not less than B,
∴ there is no triangle. [Thm. 4 Cor. 1
So, if A, B and *a* are all 90°,
then *b* is also 90°, and the triangle is biquadrantal and indeterminate. [Thm. 1 Cor. 3



(3) If B is nearer 90° than A,
then if there is a triangle ACD, c' is nearer 90° than c'',
and, in general, $c_1 + c''$ and $c_2 + c''$ both lie between 0° and 180°.
But, if $p = A$,
then $c' = 90°$, and $c_1 + c'' = c_2 + c''$,
∴ there is but one (a quadrantal) triangle.
So, if p is nearer 90° than A,
then $\sin c' > 1$, which is impossible;

$\therefore$ there is no triangle ACD, and no triangle ABC.
And $\because$ B is nearer 90° than A, [hypoth.
$\therefore$ there is no triangle when A and a are of different species.
[Thm. 4 Cor. 1

NOTE 3. If, for the given triangle, its polar be substituted, then the two sides and an angle opposite one of them are known, and Case 4 is embraced in Case 3. The required parts are the supplements of the parts found in the polar.

CASE 5. *Given the three sides*, a, b, c; then:

$\because \cos a = \cos p \cos q''$ and $\cos b = \cos p \cos q'$, [153
$\therefore \cos a : \cos b = \cos q'' : \cos q'$, [Thm. prop'n
and $\cos a + \cos b : \cos a - \cos b = \cos q'' + \cos q' : \cos q'' - \cos q'$.
But $\because \cos a + \cos b : \cos a - \cos b = -\cot \frac{1}{2}(a+b) \cot \frac{1}{2}(a-b)$, [77
and $\cos q' + \cos q'' : \cos q'' - \cos q'$
$= -\cot \frac{1}{2}(q' + q'') \cot \frac{1}{2}(q'' - q')$,
and $\because c = q' + q''$;
$\therefore \cot \frac{1}{2}(a+b) \cot \frac{1}{2}(a-b) = \cot \frac{1}{2} c \cot \frac{1}{2}(q'' - q')$, whence $\frac{1}{2}(q'' - q')$ is found.
And $\because q' = \frac{1}{2} c - \frac{1}{2}(q'' - q')$ and $q'' = \frac{1}{2} c + \frac{1}{2}(q'' - q')$,
$\therefore$ In rt. △ ACD, b and q' are known, whence A and C$'$ are found;
In rt. △ BCD, a and q'' are known, whence ∠ CBD and C$''$ are found;
$\mathrm{C} = \mathrm{C}' + \mathrm{C}''$.

NOTE 1. B = CBD or 180° − CBD, and C$''$ is positive or negative with reference to positive rotation, according as q'' is positive or negative with reference to AB.

NOTE 2. If $a + b + c =$ or $> 360°$, or if either side = or > the sum of the other two, there is no triangle; otherwise there is one triangle, and but one.

The parts are determined without ambiguity by the formulae.

CASE 6. *Given the three angles*, A, B, C; then:

$\because \cos \mathrm{A} = \cos p \sin \mathrm{C}'$, and $\cos \mathrm{B} = \cos p \sin \mathrm{C}''$, [154
$\therefore \cos \mathrm{A} : \cos \mathrm{B} = \sin \mathrm{C}' : \sin \mathrm{C}''$; [Thm. prop'n
and $\cos \mathrm{A} + \cos \mathrm{B} : \cos \mathrm{A} - \cos \mathrm{B} = \sin \mathrm{C}' + \sin \mathrm{C}'' : \sin \mathrm{C}' - \sin \mathrm{C}''$.

But $\because \cos A + \cos B : \cos A - \cos B = - \cot \frac{1}{2}(A+B) \cot \frac{1}{2}(A-B)$, [77
and $\sin C' + \sin C'' : \sin C' - \sin C'' = \tan \frac{1}{2}(C'+C'') \cot \frac{1}{2}(C'-C'')$, [72
and $\because C = C' + C''$;
$\therefore - \cot \frac{1}{2}(A+B) \cot \frac{1}{2}(A-B) = \tan \frac{1}{2} C \cot \frac{1}{2}(C'-C'')$, whence $\frac{1}{2}(C'-C'')$ is found.
And $\because C' = \frac{1}{2}C + \frac{1}{2}(C'-C'')$ and $C'' = \frac{1}{2}C - \frac{1}{2}(C'-C'')$,
$\therefore$ In rt. $\triangle$ ACD, A and C' are known, whence b and q' are found;
In rt. $\triangle$ BCD, $\angle$ CBD and C'' are known, whence a and q'' are found;
$c = q' + q''$.

NOTE 1. B = CBD or 180° − CBD, and q'' is positive or negative with reference to AB, according as C'' is positive or negative with reference to positive rotation.

NOTE 2. If $A + B + C$ does not lie between 180° and 540°, or if either angle does not exceed the difference between 180° and the sum of the other two angles, there is no triangle; otherwise there is one triangle, and but one.

The parts are determined without ambiguity by the formulae.

NOTE 3. If for the given triangle its polar be substituted, then the three sides are known, and Case 6 is embraced in Case 5. The required parts are the supplements of the parts found in the polar.

SECOND METHOD. *By means of the general properties.*

CASE 1. *Given two sides and the included angle*, for example b, c and A.

Find $\frac{1}{2}(B + C)$ and $\frac{1}{2}(B - C)$ by Thm. 12, then:

$B = \frac{1}{2}(B + C) + \frac{1}{2}(B - C)$, and $C = \frac{1}{2}(B + C) - \frac{1}{2}(B - C)$,

$\sin a = \dfrac{\sin b}{\sin B} \cdot \sin A$, whence a is found. [Thm. 2

NOTE. There is always one triangle, and but one.

The parts B and C are determined without ambiguity by the formulae, and the species of a is determined by Thm. 12, Cor.

Case 2. *Given two angles and the included side*, for example C, A and b.

Find $\frac{1}{2}(c+a)$ and $\frac{1}{2}(c-a)$ by Thm. 12; then:

$$c = \tfrac{1}{2}(c+a) + \tfrac{1}{2}(c-a), \text{ and } a = \tfrac{1}{2}(c+a) - \tfrac{1}{2}(c-a),$$

$$\sin B = \frac{\sin A}{\sin a} \sin b, \text{ whence B is found.} \qquad \text{[Thm. 2}$$

Note 1. There is always one triangle, and but one.

The parts c and a are determined without ambiguity by the formulae, and the species of b is determined by Thm. 12, Cor.

Note 2. The solution may also be obtained by applying the methods of Case 1 to the polar triangle.

Case 3. *Given two sides and an angle opposite one of them*, for example a, b and A; then:

$$\sin B = \frac{\sin A}{\sin a} \sin b, \text{ whence B is found.} \qquad \text{[Thm. 2}$$

Find c and C from the formulae of Thm. 12.

Note. There may be two triangles, one triangle or none.

If $\sin A \sin b < \sin a$, in general, there are two triangles;
for then $\sin B < 1$, and B (determined from its sine) may be either of two angles which are supplementary to each other, and the side CB may lie to the right or to the left of CD.

But this is limited by the condition that the greater angle lies opposite the greater side, and that no angle or side can exceed 180°, or be negative.

If $\sin A \sin b = \sin a$, there is one (a right) triangle;
for then $\sin B = 1$, and B is a right angle.

If $\sin A \sin b > \sin a$, there is no triangle;
for then $\sin B > 1$, which is impossible.

For detail of specific conditions the reader may consult the note to *First Method* for solving this case.

Case 4. *Given two angles and a side opposite one of them*, for example A, B and a; then:

$$\sin b = \frac{\sin a}{\sin A} \cdot \sin B, \text{ whence } b \text{ is found.} \qquad \text{[Thm. 2}$$

Find c and C from the formulae of Thm. 12.

Note 1. There may be two triangles, one triangle or none.

If $\sin a \sin B < \sin A$, in general, there are two triangles;
for then $\sin b < 1$, and b (determined from its sine) may be either of two arcs which are supplementary to each other.

But this is limited by the condition that the greater side lies opposite the greater angle, and that no side or angle can exceed 180°, or be negative.

If $\sin a \sin B = \sin A$, there is one (a quadrantal) triangle;
for then $\sin b = 1$, and b is a quadrant.

If $\sin a \sin B > \sin A$, there is no triangle;
for then $\sin b > 1$, which is impossible.

For detail of specific conditions the reader may consult the note to *First Method* for solving this case.

Note 2. The solution may also be obtained by applying the methods of Case 3 to the polar triangle.

Case 5. *Given the three sides*, a, b, c.

Apply the formulae of Thm. 5, 6 or 7.

Note 1. If either side equals or exceeds the sum of the other two, or if the sum of the three sides equals or exceeds 360°, there is no triangle; otherwise there is one triangle, and but one.

That there is a single triangle appears also from the formulae, since the half-angles computed must be each less than 90°, and but one such half-angle can be found from a given function.

Note 2. Of these formulae those of Thm. 5 use nine different logarithms, those of Thm. 6 use ten different logarithms, and those of Thm. 7 use only seven different logarithms for the computation of all the angles. Those of Thm. 7 are, therefore, generally to be preferred; they may be put in the form

$$\tan\tfrac{1}{2}A = \frac{1}{\sin(s-a)}\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}},$$

$$\tan\tfrac{1}{2}B = \frac{1}{\sin(s-b)}\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}},$$

$$\tan\tfrac{1}{2}C = \frac{1}{\sin(s-c)}\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}},$$

wherein the second factor of the right member is the same, and may be computed once for all.

NOTE 3. A complete check is afforded by either of Delambre's formulae, [165–168].

CASE 6. *Given the three angles* A, B, C.

Apply the formulae of Thm. 8, 9 or 10.

NOTE 1. If either angle does not exceed the difference between 180° and the sum of the other two, or if the sum of the angles does not lie between 180° and 540°, there is no triangle; otherwise, there is one triangle, and but one.

That there is a single solution appears also from the formulae, since the half-sides computed must be less than a quadrant, and but one such half-side can be found from a given function.

NOTE 2. Among the formulae the same choice may be made as in Case 1. The reader may transform those of Thm. 10 for convenient use.

NOTE 3. A complete check is afforded by either of Delambre's formulae, [165–168].

NOTE 4. The solution may also be obtained by applying the methods of Case 5 to the polar triangle.

The following formulae, which the reader may deduce from [159–164], are sometimes of use:

180] $$\frac{\sin s}{\sin c} = \frac{\cos\frac{1}{2}\mathrm{A}\cos\frac{1}{2}\mathrm{B}}{\sin\frac{1}{2}\mathrm{C}}, \qquad \frac{\cos \mathrm{S}}{\sin \mathrm{C}} = -\frac{\sin\frac{1}{2}a\sin\frac{1}{2}b}{\cos\frac{1}{2}c};$$

181] $$\frac{\sin(s-c)}{\sin c} = \frac{\sin\frac{1}{2}\mathrm{A}\sin\frac{1}{2}\mathrm{B}}{\sin\frac{1}{2}\mathrm{C}}, \qquad \frac{\cos(\mathrm{S}-\mathrm{C})}{\sin \mathrm{C}} = \frac{\cos\frac{1}{2}a\cos\frac{1}{2}b}{\cos\frac{1}{2}c};$$

182] $$\frac{\sin(s-a)}{\sin c} = \frac{\cos\frac{1}{2}\mathrm{A}\sin\frac{1}{2}\mathrm{B}}{\cos\frac{1}{2}\mathrm{C}}, \qquad \frac{\cos(\mathrm{S}-\mathrm{A})}{\sin \mathrm{C}} = \frac{\sin\frac{1}{2}a\cos\frac{1}{2}b}{\sin\frac{1}{2}c};$$

183] $$\frac{\sin(s-c)}{\sin(s-a)} = \frac{\tan\frac{1}{2}\mathrm{A}}{\tan\frac{1}{2}\mathrm{C}}, \qquad \frac{\cos(\mathrm{S}-\mathrm{C})}{\cos(\mathrm{S}-\mathrm{A})} = \frac{\cot\frac{1}{2}a}{\cot\frac{1}{2}c};$$

184] $$\frac{\sin(s-a)}{\sin s} = \tan\tfrac{1}{2}\mathrm{B}\tan\tfrac{1}{2}\mathrm{C}, \qquad \frac{\cos(\mathrm{S}-\mathrm{A})}{\cos \mathrm{S}} = -\cot\tfrac{1}{2}b\cot\tfrac{1}{2}c.$$

185] $$\sin(s-a)\tan\tfrac{1}{2}\mathrm{A} = \sin(s-b)\tan\tfrac{1}{2}\mathrm{B} = \sin(s-c)\tan\tfrac{1}{2}\mathrm{C}.$$ [183

§ 7. RELATIONS BETWEEN SPHERICAL AND PLANE TRIGONOMETRY.

Manifestly, when the sides of a spherical triangle subtend very small angles at the center of the sphere, the spherical triangle differs but little from a plane triangle having the same vertices; and, if the vertices be fixed in position while the center of the sphere recedes further and further away, and the radii grow longer and longer, then the spherical triangle approaches closer and closer to the plane triangle having the same vertices. Then also the small angles at the center of the sphere, which are measured by the sides of the triangle, are very nearly equivalent to their sines or tangents, and the sum of the three angles of the triangle is very nearly 180°.

If, in the several formulae of spherical trigonometry, a is substituted for $\sin a$, $\tan a$, $2 \sin \frac{1}{2} a$, ... ; p, for $\sin p$, ... ; s, for $\sin s$, ... ; and 1, for $\cos a$, ... ; then, in general, these formulae reduce to the corresponding formulae of plane trigonometry, or to mere truisms. Thus:

	Spherical.	*Plane.*	
151]	$\sin p = \sin r \sin P = \tan q \cot Q$;	$p = r \sin P = q \cot Q$ or $y = r \sin O = x \cot P$.	[I. § 7
154]	$\cos P = \cos p \sin Q = \tan q \cot r$;	$\cos P = 1 . \sin Q = q : r$ or $\cos O = \sin P = x : r$.	[I. § 7
156]	$\sin a : \sin b = \sin A : \sin B$;	$a : b = \sin A : \sin B$.	[103
159]	$\sin \frac{1}{2} A = \sqrt{\frac{\sin (s-b) \sin (s-c)}{\sin b \sin c}}$;	$\sin \frac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{bc}}$.	[109
160]	$\cos \frac{1}{2} A = \sqrt{\frac{\sin s \sin (s-a)}{\sin b \sin c}}$;	$\cos \frac{1}{2} A = \sqrt{\frac{s(s-a)}{bc}}$.	[110
161]	$\tan \frac{1}{2} A = \sqrt{\frac{\sin (s-b) \sin (s-c)}{\sin s \sin (s-a)}}$;	$\tan \frac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$.	[111

Spherical. *Plane.*

166] $\sin\tfrac{1}{2}(\mathrm{A}\sim\mathrm{B})=\dfrac{\sin\frac{1}{2}(a\sim b)}{\sin\frac{1}{2}c}\cos\tfrac{1}{2}\mathrm{C}$;

$\sin\tfrac{1}{2}(\mathrm{A}\sim\mathrm{B})=\dfrac{a\sim b}{c}\cos\tfrac{1}{2}\mathrm{C}$. [121

168] $\cos\tfrac{1}{2}(\mathrm{A}\sim\mathrm{B})=\dfrac{\sin\frac{1}{2}(a+b)}{\sin\frac{1}{2}c}\sin\tfrac{1}{2}\mathrm{C}$;

$\cos\tfrac{1}{2}(\mathrm{A}\sim\mathrm{B})=\dfrac{a+b}{c}\sin\tfrac{1}{2}\mathrm{C}$. [120

170] $\tan\tfrac{1}{2}(\mathrm{A}\sim\mathrm{B})=\dfrac{\sin\frac{1}{2}(a\sim b)}{\sin\frac{1}{2}(a+b)}\cot\tfrac{1}{2}\mathrm{C}$;

$\tan\tfrac{1}{2}(\mathrm{A}\sim\mathrm{B})=\dfrac{a\sim b}{a+b}\cot\tfrac{1}{2}\mathrm{C}$. [107

171] $\tan\tfrac{1}{2}(a+b)=\dfrac{\cos\frac{1}{2}(\mathrm{A}\sim\mathrm{B})}{\cos\frac{1}{2}(\mathrm{A}+\mathrm{B})}\tan\tfrac{1}{2}c$;

$a+b=\dfrac{\cos\frac{1}{2}(\mathrm{A}\sim\mathrm{B})}{\cos\frac{1}{2}(\mathrm{A}+\mathrm{B})}\cdot c$. [120

172] $\tan\tfrac{1}{2}(a\sim b)=\dfrac{\sin\frac{1}{2}(\mathrm{A}\sim\mathrm{B})}{\sin\frac{1}{2}(\mathrm{A}+\mathrm{B})}\tan\tfrac{1}{2}c$;

$a\sim b=\dfrac{\sin\frac{1}{2}(\mathrm{A}\sim\mathrm{B})}{\sin\frac{1}{2}(\mathrm{A}+\mathrm{B})}\cdot c$. [121

* Some correspondences between the formulae of plane and of spherical trigonometry appear only when functions of the sides, of their half-sum, etc., are represented by taking two or more terms of the series [94–99] instead of using the first term alone as above. Thus:

Formula $\cos r=\cos p\cos q$ [153

becomes a mere truism if each cosine be represented by 1, its limit when p, q and r become indefinitely small; but if the values

$\cos p=1-\tfrac{1}{2}p^2+\cdots\cdots$, $\cos q=1-\tfrac{1}{2}q^2+\cdots\cdots$, $\cos r=1-\tfrac{1}{2}r^2+\cdots\cdots$, [95

be substituted in [153] they give

$$\begin{aligned}1-\tfrac{1}{2}r^2+\cdots\cdots&=(1-\tfrac{1}{2}p^2+\cdots\cdots)(1-\tfrac{1}{2}q^2+\cdots\cdots)\\&=1-\tfrac{1}{2}p^2-\tfrac{1}{2}q^2+\cdots\cdots;\end{aligned}$$

$\therefore$ $p^2 + q^2 = r^2 \pm$ terms of higher degree, whose ratios to p^2, q^2 and r^2 have the limit 0 when p, q and r become indefinitely small;

i.e., $p^2 + q^2$, or $x^2 + y^2$, $= r^2$, whence $\sin^2\theta + \cos^2\theta = 1$;

$\therefore$ [153] corresponds to [36].

So, writing b for $\sin b$, c for $\sin c$, $1 - \frac{1}{2}a^2 + \cdots\cdots$ for $\cos a$, $\cdots\cdots$, then the formula

$\cos a = \cos b \cos c + \sin b \sin c \cos \mathrm{A}$ [157

gives $a^2 = b^2 + c^2 - 2bc \cos \mathrm{A}$. [108

So, writing two terms of [94] for $\sin b$, $\cdots\cdots$, and three terms of [95] for $\cos a$, $\cdots\cdots$, then the formula

$\cos a = \cos b \cos c + \sin b \sin c \cos \mathrm{A}$ [157

gives $bc(\cos \mathrm{A}' - \cos \mathrm{A}) = \frac{1}{12}(a^2b^2 + b^2c^2 + c^2a^2) - \frac{1}{24}(a^4 + b^4 + c^4)$ $\pm$ terms whose limiting ratios to these terms, when a, b and c become indefinitely small, are 0;

and so for $ca(\cos \mathrm{B}' - \cos \mathrm{B})$, and $ab(\cos \mathrm{C}' - \cos \mathrm{C})$. [symmetry

$\therefore$ $bc(\cos \mathrm{A}' - \cos \mathrm{A}) = ca(\cos \mathrm{B}' - \cos \mathrm{B}) = ab(\cos \mathrm{C}' - \cos \mathrm{C})$ approximately, wherein $\cos \mathrm{A}' = (b^2 + c^2 - a^2) : 2bc$, $\cdots\cdots$,

and A', B' and C' are the angles of the plane triangle whose sides a', b' and c' are respectively equal to the arcs a, b and c.

But $\because$ $\cos \mathrm{A}' - \cos \mathrm{A} = 2 \sin \frac{1}{2}(\mathrm{A} - \mathrm{A}') \sin \frac{1}{2}(\mathrm{A} + \mathrm{A}')$
$= (\mathrm{A} - \mathrm{A}') \sin \mathrm{A}'$, very nearly,

and so for $\cos \mathrm{B}' - \cos \mathrm{B}$ and $\cos \mathrm{C}' - \cos \mathrm{C}$, [symmetry

$\therefore$ $bc(\mathrm{A} - \mathrm{A}') \sin \mathrm{A}' = ca(\mathrm{B} - \mathrm{B}') \sin \mathrm{B}' = ab(\mathrm{C} - \mathrm{C}') \sin \mathrm{C}'$,

$\therefore$ $\mathrm{A} - \mathrm{A}' = \mathrm{B} - \mathrm{B}' = \mathrm{C} - \mathrm{C}'$, very nearly; [156

186] $\therefore$ each angle of sph. $\triangle$ABC exceeds the corresponding angle of pl. $\triangle$A′B′C′ by one-third of the spherical excess, $\mathrm{A} + \mathrm{B} + \mathrm{C} - 180°$; which is *Legendre's* theorem.

§ 8. EXERCISES.

Solve the spherical right triangles, given:

1. $p = 116°$, $q = 16°$, $\mathrm{R} = 90°$.
2. $r = 140°$, $p = 20°$, $\mathrm{R} = 90°$.
3. $\mathrm{P} = 80° 10'$, $q = 155° 46'$, $\mathrm{R} = 90°$.

4. $P = 100^\circ$, $p = 112^\circ$, $R = 90^\circ$.
5. $r = 120^\circ$, $P = 120^\circ$, $R = 90^\circ$.
6. $P = 60^\circ 47'$, $Q = 57^\circ 16'$, $R = 90^\circ$.
7. $r = 140^\circ$, $p = 140^\circ$, $R = 90^\circ$.
8. $r = 120^\circ$, $P = 90^\circ$, $R = 90^\circ$.

Solve the quadrantal triangles, given:

9. $A = 80^\circ$, $a = 90^\circ$, $b = 37^\circ$.
10. $B = 50^\circ$, $b = 130^\circ$, $c = 90^\circ$.

Solve the isosceles triangles, given:

11. $a = 70^\circ$, $b = 70^\circ$, $A = 30^\circ$.
12. $a = 30^\circ$, $A = 70^\circ$, $B = 70^\circ$.
13. $a = 119^\circ$, $b = 119^\circ$, $C = 85^\circ$.

Solve the oblique triangles [both methods], given:

14. $b = 98^\circ 12'$, $c = 80^\circ 35'$, $A = 10^\circ 16'$.
15. $A = 135^\circ 15'$, $C = 50^\circ 30'$, $b = 69^\circ 34'$.
16. $a = 40^\circ 16'$, $b = 47^\circ 14'$, $A = 52^\circ 30'$.
17. $a = 120^\circ$, $b = 70^\circ$, $A = 130^\circ$.
18. $a = 40^\circ$, $b = 50^\circ$, $A = 50^\circ$.
19. $A = 132^\circ 16'$, $B = 139^\circ 44'$, $a = 127^\circ 30'$.
20. $A = 110^\circ$, $B = 60^\circ$, $a = 50^\circ$.
21. $A = 70^\circ$, $B = 120^\circ$, $a = 80^\circ$.
22. $a = 100^\circ$, $b = 50^\circ$, $c = 60^\circ$.
23. $A = 120^\circ$, $B = 130^\circ$, $C = 80^\circ$.

24. In astronomy the *altitude* of a heavenly body is its angular elevation above the horizon, and its *azimuth* is its angular distance west from the south point. What is the angular distance between the moon, alt. 40°, az. 25° W., and Venus, alt. 24°, az. 110° W.?

25. In navigation the shortest distance from port to port is the arc of a great circle. Find the course and distance from San Francisco, lat. 37° 48′ N., long. 122° 25′ W., to Cape of Good Hope, lat. 33° 56′ S., long. 18° 29′ E., no allowance being made for intervening lands.